ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Unsupervised and reinforcement learning in neural networks
Week 8

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Solution: Restricted Boltzmann Machines

1 Inference

Step 1: $P(x_1) = \sum_{x_2} P(x_1, x_2) = \sum_{x_2} A^{-1} f(x_1) f(x_2) = f(x_1) A^{-1} \sum_{x_2} f(x_2) \propto f(x_1)$. Analogously, $P(x_2) \propto f(x_2)$, with normalization constant $\sum_{x_2} f(x_2)$.

$P(x_1, x_2) = A^{-1} f(x_1) f(x_2) \propto P(x_1)P(x_2) \implies P(x_1, x_2) = P(x_1)P(x_2)$.

Step 2: Separating y_i terms from the others,

$$P(\bar{y}|\bar{x}^\mu) = \frac{1}{ZP(\bar{x}^\mu)} e^{\sum_i a_i x_i} \prod_j e^{b_j y_j + \sum_i w_{i,j} x_i y_j} = A \prod_j f(y_j). \tag{1}$$

This shows the conditional distribution is separable, so they are independent, $P(\bar{y}|\bar{x}^\mu) = \prod_j P(y_j|\bar{x}^\mu)$.

Step 3: We have that $P(y_m|\bar{x}^\mu) \propto f(y_j) = e^{b_m y_m + \sum_i w_{i,m} x_i y_m}$, so

$$\frac{P(y_k = 1|\bar{x}^\mu)}{P(y_k = 0|\bar{x}^\mu)} = \frac{\frac{1}{A} e^{b_k 1 + \sum_i w_{i,k} x_i 1}}{\frac{1}{A} e^{b_k 0 + \sum_i w_{i,k} x_i 0}} = e^{b_k + \sum_i w_{i,k} x_i^\mu}. \tag{2}$$

Step 4: Let $P_1 = P(y_k = 1|\bar{x}^\mu)$ and $P_0 = P(y_k = 0|\bar{x}^\mu)$. We have that $P_1 + P_0 = 1$ and $P_1/P_0 = e^{b_k + \sum_i w_{i,k} x_i^\mu}$. Solving on P_1, we find $P_1 = \frac{1}{1 + e^{-(b_k + \sum_i w_{i,k} x_i^\mu)}} = \sigma(b_k + \sum_i w_{i,k} x_i^\mu)$.

By symmetry, one finds that $P(x_m = 1|\bar{y}) = \sigma(a_m + \sum_j w_{m,j} y_j)$.

This means inference is easy, given a data sample you can estimate each hidden unit independently, with an sigmoidal activation probability. Symmetrically, visible units are independent given the hidden vector.

2 Learning

$$logP(\bar{x}, \bar{y}) = \sum_i a_i x_i + \sum_j b_j y_j + \sum_{i,j} w_{i,j} x_i y_j - logZ \tag{3}$$

$$\frac{\partial logP(\bar{x}^\mu, \bar{y}^\mu)}{\partial w_{i,j}} = x_i^\mu y_j^\mu - \frac{\partial logZ}{\partial w_{i,j}} \tag{4}$$

$$\frac{\partial logZ}{\partial w_{i,j}} = \frac{1}{Z} \sum_{\{\tilde{x},\tilde{y}\}} \tilde{x}_i \tilde{y}_j e^{-E(\tilde{x},\tilde{y})} = \sum_{\{\tilde{x},\tilde{y}\}} \tilde{x}_i \tilde{y}_j \frac{e^{-E(\tilde{x},\tilde{y})}}{Z} = \sum_{\{\tilde{x},\tilde{y}\}} \tilde{x}_i \tilde{y}_j p(\tilde{x}, \tilde{y}). \tag{5}$$