ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Unsupervised and reinforcement learning in neural networks
Week 1

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Revision – Linear Algebra

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These exercises will be done during the tutorial on Wednesday, 17.09.2014.

Exercise 1: Eigenvalue Problems

Determine the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$D = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad E = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

Are the eigenvectors orthogonal? Are they unique?

Exercise 2: From Data to the Covariance Matrix

Calculate the covariance matrix given the following datapoints:
$[1, 1]$, $[2, 1]$, $[1, 2]$, $[2, 2]$, $[-1, -1]$, $[-2, -1]$, $[-1, -2]$, $[-2, -2]$

Hint: Plot the data and identify potential symmetries that might help you compute the covariance matrix.

Exercise 3: Properties of Covariance Matrices

Let $\mathbf{x}^\mu \in \mathbb{R}^N$, $\mu = 1, ..., M$ be a set of data points and $\langle \mathbf{x} \rangle := \frac{1}{M}\sum_{\mu=1}^{M} \mathbf{x}^\mu$ their mean value.

1. Show that the covariance matrix
$$\mathbf{C} := \frac{1}{M}\sum_{\mu=1}^{M}(\mathbf{x}^\mu - \langle \mathbf{x} \rangle)(\mathbf{x}^\mu - \langle \mathbf{x} \rangle)^T \tag{1}$$
is symmetric and positive semi-definite. **Reminder:** A matrix $\mathbf{A}$ is called positive semi-definite if $\mathbf{v}^T\mathbf{A}\mathbf{v} \geq 0$ for all vectors $\mathbf{v}$.

2. Let $\mathbf{n} \in \mathbb{R}^N$ be normalized, i.e., $||\mathbf{n}|| = 1$. What is the intuitive interpretation of the quantity $\mathbf{n}^T\mathbf{C}\mathbf{n}$?

Exercise 4: Projectors

1. Let $\mathbf{n} \in \mathbb{R}^N$ be a normalized vector. Describe the effect of the matrix $\mathbf{A} = \mathbf{n}\mathbf{n}^T$ on an arbitrary vector $\mathbf{x}$.
2. A matrix is a projector if and only if $P^2 = P$. What is the intuition for this condition?
3. Show that $\mathbf{A}$ is a projector.
4. Construct a matrix such that the vector $\mathbf{x}_{\|} = \mathbf{P}\mathbf{x}$ lies in the plane that is spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$. Note that $\mathbf{a}$ and $\mathbf{b}$ may not be normalized and orthogonal.
5. Which of the matrices in Exercise 1 are projectors?