ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Unsupervised and reinforcement learning in neural networks

Week 3

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Whitening, Decorrelation & Independence

Exercise 1

1.1. In class: Prove that if x_1 and x_2 are statistically independent, they are also uncorrelated:

$$p(x_1, x_2) = p_1(x_1)p_2(x_2) \Rightarrow \langle (x_1 - \langle x_1 \rangle)(x_2 - \langle x_2 \rangle) \rangle = 0.$$

1.2. For N-dimensional data, the Gaussian distribution has the form:

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N det(C)}} \exp\left(-\frac{1}{2}(\vec{x} - \vec{\mu})^T C^{-1} (\vec{x} - \vec{\mu}) \right),$$

where and $\vec{\mu}$ is the mean of the data, C their covariance matrix, and $det(C)$ its determinant. Prove for Gaussian random variables: If all pairs x_i, x_j of variables are decorrelated, the variables are statistically independent:

$$\forall i, j : \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle = 0 \Rightarrow p(\vec{x}) = \prod_i p_i(x_i).$$

Exercise 2

Assume that a dataset $\{\vec{x}^\mu\}$ is whitened, that is, each component has zero mean and unit variance, and the components are pairwise decorrelated. Show that any rotation $\{\vec{y}^\mu = R\vec{x}^\mu\}$ of the data is also whitened.
Hint: R is a rotation matrix, iff $RR^T = E$, E being the identity matrix.

Exercise 3

Prove: Given two functions h_1, h_2 and two independent random variables x_1 and x_2, the expectation value of the product of h_1 and h_2 factorizes in the product of the expectation values :

$$p(x_1, x_2) = p_1(x_1)p_2(x_2) \Rightarrow E\{h_1(x_1)h_2(x_2)\} = E\{h_1(x_1)\}E\{h_2(x_2)\}$$

Exercise 4

Variance is defined as $var(x) = E\{x^2\} - E\{x\}^2$, kurtosis as $\kappa(y) = E\{x^4\} - 3(E\{x^2\})^2$. Calculate variance and kurtosis of

4.1 the Gaussian distribution:

$$p(x) = \sqrt{\frac{a}{\pi}} \exp\left(-ax^2\right).$$

Hint: $x^2 \exp(-ax^2) = -\frac{d}{da} \exp(-ax^2)$.

4.2 the uniform distribution:

$$p(x) = \begin{cases} \frac{1}{2\sqrt{3}} & \text{if } |x| \leq \sqrt{3} \\ 0 & \text{otherwise.} \end{cases}$$

4.3 the exponential distribution (Laplace distribution):

$$p(x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|).$$