

Solution 1: spin detection

a)

Nucleus	A	Z	N	Predicted I	Detectable
^4He	4	2	2	0	No
^6Li	6	3	3	Integer	Yes
^7Li	7	3	4	$n/2(n \text{ odd})$	Yes

Group1: Nuclei with both Z and N even. In such nuclei all proton spins are paired and all neutron spins are paired, resulting in a net nuclear spin of zero ($I=0$).

Group2: Nuclei with both Z and N odd. Such nuclei must have an odd number of unpaired proton ($I=1/2$) spins, and an odd number of unpaired neutron ($I=1/2$) spins, so the net magnetic spin must be a nonzero integer.

Group3: Nuclei with even Z and odd N or odd Z and even N. These nuclei must have an even number of proton spins and an odd number of unpaired neutron spins, or vice versa. Therefore, the net magnetic spin is an odd integer multiple of $1/2$.

b)

$$\Delta E = \frac{\gamma \hbar B_0}{2\pi} = \frac{(267.512 \times 10^6 \text{ rad} \cdot \text{T}^{-1} \cdot \text{s}^{-1})(6.63 \times 10^{-34} \text{ J} \cdot \text{s})(5.87 \text{ T})}{2 \cdot 3.14 \text{ rad}} = 1.66 \times 10^{-25} \text{ J}$$

$$\text{For } ^{13}\text{C}, \Delta E = 4.18 \times 10^{-26} \text{ J}$$

c)

Using Boltzmann equation (describing energy state population), the ratio between the populations of the two spin states is given by:

$$\text{For proton, } \frac{P_{(m=-1/2)}}{P_{(m=+1/2)}} = \exp\left(-\frac{\Delta E}{kT}\right) = \exp\left(-\frac{1.66 \times 10^{-25} \text{ J}}{1.381 \times 10^{-23} \text{ JK}^{-1} \times 298 \text{ K}}\right) = 0.99996$$

Since there are only two spin states,

$$P_{(m=-1/2)} = P_{\text{tot}} - P_{(m=+1/2)}$$

We thus obtain:

$$P_{(m=-1/2)} = 0.49999 \cdot P_{\text{tot}}$$

$$P_{(m=1/2)} = 0.50001 \cdot P_{\text{tot}}$$

This means that the spin states have 0.004% of population difference.

Using the same equations for ^{13}C , we obtain :

$$P_{(m=-1/2)} = 0.4999975 \cdot P_{\text{tot}}$$

$$P_{(m=1/2)} = 0.5000025 \cdot P_{\text{tot}}$$

The spin states have 0.001% of population difference.

It means that, at same temperature, protons spins have a bigger population difference than carbon (4 times bigger). Moreover, magnetization is a macroscopic value and thus depends on the total number of spins in the sample. As there are many more ^1H than ^{13}C (99.98% vs. 1.1% natural abundance) in living tissues, proton magnetization is much higher than carbon due to these two factors.

d)

Proton is used for imaging because its magnetization (hence its signal) is higher compared to the other nuclei. Its magnetization is the highest one because of:

- Its big natural abundance in tissues (water!)
- Its high gyromagnetic ratio (which means bigger difference in population of energy levels and thus higher magnetization)

e)

Glucose molecule has 6 carbon nuclei and 12 protons. Moreover ^{13}C has a natural abundance of 1.1%. Calculating the ratio between detectable carbons and protons, we obtain :

$$N_{^{13}\text{C}}/N_{^1\text{H}} = 0.011 \cdot 0.5 = 0.0055 = 0.55\%$$

Moreover using above calculations for differences in magnetization between proton and carbon nuclei, we can calculate the ratios between both macroscopic magnetizations :

$$M_{^{13}\text{C}}/M_{^1\text{H}} = 0.55\% \cdot 0.25 = 0.14\%$$

The carbon signal is really weak compared to the proton one, but can be of great interest, that's why ways of increasing its MR signal have to be used (enrichment, techniques allowing to increase the signal...).

Solution 2: B_1 field and radiofrequency

- a) From slide (8-15), we find the precession equation of the magnetization originating from proton spins around a magnetic field B_1 :

$$\frac{d\vec{M}}{dt} = \gamma_{1H} \vec{M} \times \vec{B}_1$$

This is the equation of a rotation around \vec{B}_1 with a frequency $\gamma_{1H} \vec{B}_1$ [MHz]. The period p is thus $p = \frac{1}{\gamma_{1H} B_1}$. We

want to have a 90° rotation after 1ms, which means a period of 4ms. We can then derive the necessary B_1 field:

$$B_1 = \frac{1}{\gamma_{1H} p} = 5.9 \mu T \quad (\text{you find } \gamma_{1H} \text{ on slide 8-8})$$

b) With the same argument, we find

$$B_1 = \frac{1}{\gamma_{13C} p} = 23.3 \mu T$$

c) We decide to have the magnetization in both cases (1H and ^{13}C) in 1ms at 90°. So, we have for both cases a frequency of $1/4\text{ms}=250\text{Hz}$. This frequency has no direct relation with the precession frequency of each nucleus around B_0 .

d) To be on resonance, B_1 must be stationary in the rotating frame turning at the Larmor frequency. For 1H , the Larmor frequency at 9.4T is given by: $f = \gamma_{1H} B_0 = 400.25 \text{ MHz}$.

e) The related wavelength is given by: $\lambda = \frac{c}{f} = 75 \text{ cm}$.

The corresponding photon energy is given by the Planck's relation: $E = hf = 2.65 \cdot 10^{-25} \text{ J} = 1.66 \cdot 10^{-6} \text{ eV}$

We are well below the ionizing radiation limit (13.6eV).

In X-ray imaging, we are dealing with photon energies bigger than 1 keV and wavelength below 1nm.

Solution 3: Rotating frame and effective field

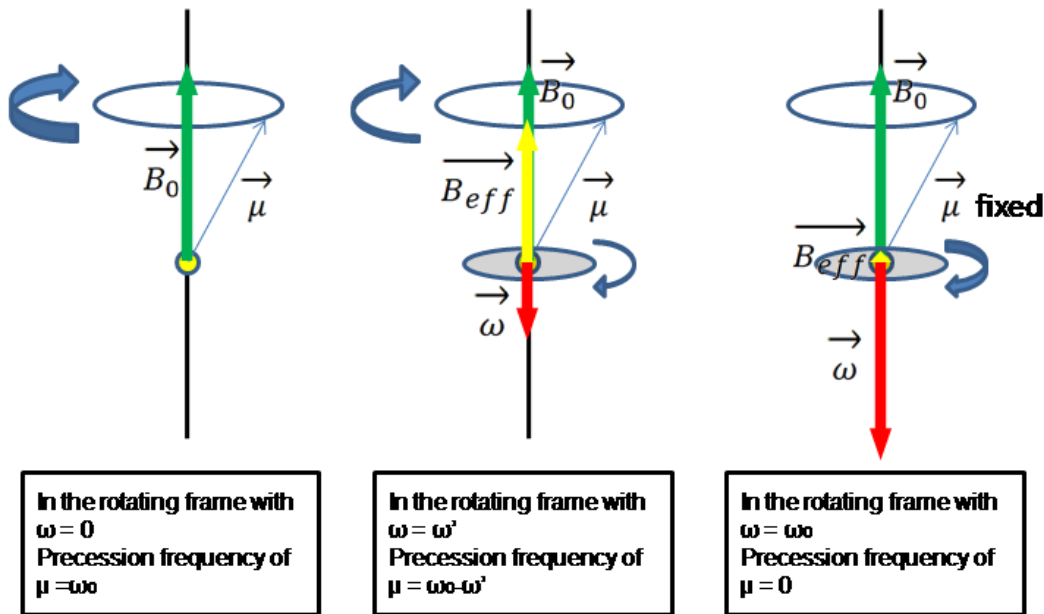
$$a) \quad \frac{d\vec{m}(t)}{dt} = \gamma \vec{m}(t) \times \vec{B}_0 \quad \text{and} \quad \left[\frac{d\vec{A}(t)}{dt} \right]_{static} = \left[\frac{\partial \vec{A}(t)}{\partial t} + \vec{\omega} \times \vec{A}(t) \right]_{rotating}$$

$$\text{so:} \quad \frac{d\vec{m}(t)}{dt} = \frac{\partial \vec{m}(t)}{\partial t} + \vec{\omega} \times \vec{m}(t)$$

$$= \gamma \vec{m}(t) \times \vec{B}_0 \quad \text{from the precession law in the lab frame}$$

$$\begin{aligned} \frac{\partial \vec{m}(t)}{\partial t} &= \gamma \vec{m}(t) \times \vec{B}_0 - \vec{\omega} \times \vec{m}(t) = \gamma \vec{m}(t) \times \vec{B}_0 + \vec{m}(t) \times \vec{\omega}(t) = \gamma \vec{m}(t) \times \left[\vec{B}_0 + \frac{\vec{\omega}(t)}{\gamma} \right] \\ &= \gamma \vec{m}(t) \times \vec{B}_{eff} \end{aligned}$$

b)



c) $\vec{B}(t) = \vec{B}_0 + \vec{B}_1(t)$

$$\vec{B}_{eff} = \left(\vec{B}_0 + \frac{\vec{\omega}}{\gamma} \right) + \vec{B}_1(t)$$

$$B_{eff} = \|\vec{B}_{eff}\| = \left[\left(\vec{B}_0 + \frac{\vec{\omega}}{\gamma} \right)^2 + \vec{B}_1^2 \right]^{\frac{1}{2}}$$

With $\omega_1 = -\gamma B_1$

$$B_{eff} = \frac{1}{\gamma} \left[(\omega - \omega_0)^2 + \omega_1^2 \right]^{\frac{1}{2}}$$

$$\vec{B}_{eff} = \frac{1}{\gamma} \left[(\omega - \omega_0) \vec{e}_z + \omega_1 \vec{e}'_x \right]$$

So,

$$\tan \theta = \frac{\omega_1}{\omega - \omega_0} = \frac{-B_1}{B_0 + \frac{\omega}{\gamma}}$$

d) If $|\omega - \omega_0| \gg |\omega_1|$ so \vec{B}_{eff} almost aligned to \vec{B}_0

In opposite, if $|\omega - \omega_0| \ll |\omega_1|$ so \vec{B}_{eff} will create a precession around the x' axis (axis of \vec{B}_1). We are on resonance.

e)

