4

Random Regular Graphs

4.1 Introduction

Another model for random graphs is the random regular graph $G(n, r)$, in which every vertex has degree r.

Definition 4.1 (Random regular graph)**.** *Consider the set $\mathcal{G}(n, r)$ of all labeled r-regular graphs of size n with nr even, i.e., the set of labeled (simple) graphs with vertex label set $[n]$ and constant degree r. Then the* random regular graph $G(n, r)$ *is a uniform random element of $\mathcal{G}(n, r)$.*[1]

In contrast to the previous model $G(n, p)$, the existence of different edges is not independent, and this leads, not surprisingly, to some additional difficulties in the analysis. Even defining the probability space is not as straightforward as before: we would like to assign the same probability to every labelled graph over vertex set $[n] = \{1, 2 \ldots, n\}$ with degree r. In order to build up to this, we will first analyze a relaxation of this model in which we sample from a larger class of graphs $G^\star(n, r)$, by allowing for the possibility of loops and multiple edges.

[1] We observe that $G(n, r)$ has n vertices and $rn/2$ edges, which is why rn must always be even.

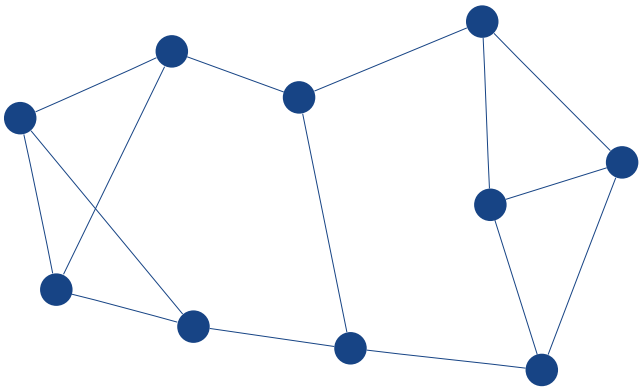

Figure 4.1: An instance of $G(10, 3)$.

4.2 Preliminaries

We need a few tools from combinatorial analysis which are well suited to the study of random regular graphs $G(n, r)$ and their extensions $G^\star(n, r)$. Let

$$(X)_m = X(X-1)(X-2)\cdots(X-m+1).$$

The mth *factorial moment* of a random variable X is

$$\mathbb{E}\left[(X)_m\right] = \mathbb{E}\left[X(X-1)(X-2)\cdots(X-m+1)\right]. \tag{4.1}$$

For instance, if $X \sim \text{Poisson}(\lambda)$, then

$$\mathbb{E}\left[(X)_m\right] = \lambda^m. \tag{4.2}$$

As a result, given a random vector $(X_1, \ldots, X_l)$, where the $X_i \sim \text{Poisson}(\lambda_i)$ are independent, we have that

$$\mathbb{E}\left[(X_1)_{m_1} \cdots (X_l)_{m_l}\right] = \lambda_1^{m_1} \cdots \lambda_l^{m_l}.$$

Moreover, we will use (without proof – see [] for a proof) the following converse convergence result when the random vector depends on $n \in \mathbb{N}$.

Theorem 4.1 (Method of factorial moments for Poisson random variables)**.** *Let $(X_{n1}, \ldots, X_{nl})$ be a random vector, where $l \geq 1$ is fixed. If $\lambda_1, \ldots, \lambda_l \geq 0$ are such that for $n \to \infty$, $\mathbb{E}\left[(X_{n1})_{m_1} \cdots (X_{nl})_{m_l}\right] \to \lambda_1^{m_1} \cdots \lambda_l^{m_l}$ for every $m_1, \ldots, m_l \geq 0$, then $(X_{n1}, \ldots, X_{nl}) \to (Z_1, \ldots, Z_l)$, where $Z_i \sim$ Poisson(λ_i) are independent.*

4.3 The matching model $G^\star(n, r)$

Consider the set of *stubs* $S = [n] \times [r]$; each stub is one endpoint of a potential edge. We will again assume nr is even. To generate a random regular multigraph $G^\star(n, r)$, we first generate a *matching* M (also called a *pairing*) of the nr stubs, which results into $nr/2$ edges. More formally, a matching is a function $M : S \to S$ such that if $M(s_1) = s_2$ then $M(s_2) = s_1$ for all stubs $s_1 = (u, e)$ and $s_2 = (v, f) \in S$. Then, the edge set of the graph $G^\star(n, r)$ given by the matching M as follows:

$$E = \{(u, v) : \exists\, e, f \text{ such that } (u, e) = M(v, f)\}.$$

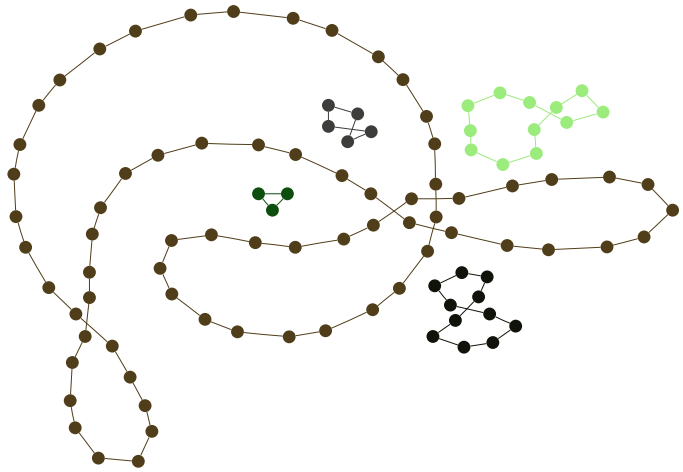

Figure 4.2: An instance of $G(100, 2)$. The graph $G(n, 2)$ is *a.a.s.* not connected (we do not prove this here), and is the union of several disjoint cycles.

Note that $G^\star(n, r)$ may not be a simple graph, because it could have self-loops if the matching has $M(u, e) = (u, f)$ for some $e \neq f \in [k]$, or it could have multiple edges if the matching has $M(u, e_1) = (v, f_1)$ *and* $M(u, e_2) = (v, f_2)$ for some $e_1 \neq e_2$ and $f_1 \neq f_2 \in S$.

This matching model is interesting because

1. If we condition on the random matching being a simple graph, then the graph is sampled uniformly from $\mathcal{G}(n, r)$,

2. the probability that the sampled graph is simple is bounded away from zero and hence a property that holds *a.a.s.* for $G^\star(n, r)$ also holds *a.a.s.* for $G(n, r)$, and

3. it is often easier to prove many properties of interest in $G^\star(n, r)$ than in $G(n, r)$ directly.

We will sketch the proof of the claim 1 here: Note that there are $(nr - 1)!! \stackrel{\text{def}}{=} (nr - 1)(nr - 3) \cdots 3$ distinct matchings. If we condition on $G^\star$ being a simple graph, then each element of $\mathcal{G}(n, r)$ is equally probable. This is because each element of $\mathcal{G}(n, r)$ corresponds to the same number of distinct configurations, namely $(r!)^n$ each. Hence the first property holds. Note that this holds only conditional on $G^\star$ being simple; unconditionally, graphs with loops and/or multiple edges appear with smaller probability than a simple graph.

In the next subsection we will illustrate claim 3 by showing how to prove a property of interest in $G^\star(n, r)$ (where no clear approach for proving in $G(n, r)$ is present); we will then leverage this result to prove claim 2 in the following subsection.

4.4 Appearance of a fixed subgraph in $G^\star(n, r)$

Let the family of random variables $(Z_k)_{k=1}^{\infty}$ denote the number of cycles of size k in $G^\star(n, r)$. In particular, Z_1 is the number of self-loops (1-cycles), and Z_2 is the number of pairs of edges between the same vertices (2-cycles).

Theorem 4.2 (Convergence in distribution of the number of all cycles in $G^\star(n,r)$)**.** *The random variables (Z_k) for $k \in \mathbb{Z}^+$ converge in distribution to a collection of independent random variables, with $Z_k \to Poisson(^{(r-1)^k}/_{2k})$.*

Proof:

Consider a sample of $G^\star(n,r)$ and consider the matching M of stubs S that produced it. Note that a cycle of size k in $G^\star$ is formed by $2k$ stubs (2 for each vertex in the cycle) that are matched appropriately. While many sets of stubs could satisfy this (in fact, $\binom{}{k,2}$ for each vertex), given a *specific* pair of stubs we call the edge it produces a *labeled edge*, and consider the *labeled cycle* that gives rise to a given cycle in $G^\star(n,k)$. We will use labeled cycles to compute the moments of Z_k, and then apply the method of moments to establish convergence in distribution.

First we need the probability p_k that a set of k labeled edges is in a random matching. This is

$$p_k = \frac{(rn-2k-1)!!}{(rn-1)!!} = \frac{1}{(rn-1)(rn-3)\cdots(rn-2k+1)}, \tag{4.3}$$

because each labeled edge blocks two stubs, which leaves $(rn-2k-1)!!$ configurations for the remaining stubs once the k pairs are fixed.

Expected number of k-cycles. We count the number of ways a k-cycle can appear in $G^\star$. As in Theorem 3.6, the number of distinct vertex-labeled k-cycles is

$$\left(\begin{array}{c} n \\ k \end{array} \right) \frac{k!}{2k}, \tag{4.4}$$

where $2k$ is the size of the automorphism group of the k-cycle.

The k-cycle obtained is a sequence of k edges $((v_i,q_i),(v_{i+1},q_{i+1}))$, with $1 \le i \le k$ (and indices taken modulo k) between distinct vertices. Hence, it is defined by a sequence of k distinct vertices $v_1,\ldots,v_k$, and for each vertex there are two distinct labels $p_i,q_i \in [r]$ in the matching. Hence the total number of distinct k-cycles is

$$\left(\begin{array}{c} n \\ k \end{array} \right) \frac{k!}{2k}(r(r-1))^k \sim \frac{(nr(r-1))^k}{2k}. \tag{4.5}$$

For large n we also have $p_k \sim (rn)^{-k}$, and therefore $\mathbb{E}[Z_k] \sim (r-1)^k/2k$.

Expected number of other graphs H. Note that a similar argument shows that the expected number of copies of a graph is in general $\Theta(n^{v(H)-e(H)})$. We will study higher-order moments next, which amounts to counting the number of copies of graphs H where H is the union of several cycles. Note that if all cycles are are disjoint, then $v(H) = e(H)$. Otherwise, H contains at least one component which is the union of several intersecting cycles; H then has $v(H) < e(H)$.

Second factorial moment. Before studying higher-order joint moments in their full generality, we consider the second factorial moment $\mathbb{E}[(Z_k)_2]$, i.e., $(Z_k)_2$ is the number of ordered pairs of distinct k-cycles in G. We can express this number as a sum of two terms S_0 and $S_>$, where S_0 counts the number of ordered pairs of two distinct *disjoint* k-cycles, and where $S_>$ counts the number of ordered pairs of two intersecting k-cycles. We now show that S_0 asymptotically dominates.

Similar to Theorem 3.6, we can express $S_>$ as a sum of terms $S_{e,f}$ according to the number of vertices i and edges j in the intersection between the two k-cycles. Obviously, the number of terms does not depend on n.

Each $S_{e,f}$ counts the number of copies of an unlabeled graph $H_{e,f}$, which is the union of two intersecting k-cycles, and thus $v(H) < e(H)$. Therefore, $S_>$ is $O(n^{v(H)-e(H)}) = o(1)$.

To compute S_0,

$$\left(\begin{array}{c} n \\ k \end{array} \right) \frac{k!}{2k} \left(\begin{array}{c} n-k \\ k \end{array} \right) \frac{k!}{2k} (r(r-1))^{2k} \sim \left(\frac{n^k r^k (r-1)^k}{2k} \right)^2 . \tag{4.6}$$

Combining this with $p_k \sim (rn)^{-k}$, we obtain, as needed, $\mathbb{E}\left[(Z_k)_2\right] \to \lambda_k^2$.

Higher-order moments of number of k-cycles. We now generalize this argument to larger cycles and joint moments, of the form

$$\mathbb{E}\left[(Z_1)_{m_1} (Z_2)_{m_2} \cdots (Z_l)_{m_l}\right]. \tag{4.7}$$

Now H denotes an unlabeled graph resulting from the union of m_1 1-cycles, m_2 2-cycles, etc. A similar argument as before shows that all the terms corresponding to H where not all cycles are disjoint go to zero. The sum S_0 is easily shown to factor, so that

$$\mathbb{E}\left[(Z_1)_{m_1} (Z_2)_{m_2} \cdots (Z_l)_{m_l}\right] \to \lambda_1^{m_1} \cdots \lambda_l^{m_l}. \tag{4.8}$$

Theorem 4.1 shows the result.

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4.5 The random regular graph $G(n, r)$

We can now go back to the original model $G(n, r)$. Consider once again the same random variables $(Z_k)_{k=1}^{\infty}$, the number of cycles of size k, but in $G(n, r)$ instead of in $G^\star(n, r)$. Obviously, this forces $Z_1 = Z_2 = 0$. The following theorem is then a direct consequence of Theorem 4.2 attianed by conditioning on $Z_1 = Z_2 = 0$.

Corollary 4.1 (Convergence in distribution of number of all cycles in $G(n, r)$)**.** *The random variables $(Z_k), k \geq 3$ converge in distribution to a collection of independent random variables, with $Z_k \to$ Poisson$((r-1)^k/2k)$.*

We can now show that the probability that $G^\star(n, r)$ is simple is bounded away from zero.

Theorem 4.3 (Probability that random regular multigraph is simple)**.** $\mathbb{P}\{G^\star(n, r)$ *is simple*$\} \to \exp\left(-\frac{r^2-1}{4}\right)$.

Proof:

$\mathbb{P}\{G^\star(n, r)$ is simple$\} = \mathbb{P}\{Z_1 = Z_2 = 0\}$.

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This gives an efficient way of generating $G(n, r)$: simply generate random matchings of stubs until a simple graph is found. Unfortunately, however, the probability of success decreases quite quickly with r.

Theorem 4.4 (Properties of $G^\star(n, r)$ are properties of $G(n, r)$ almost sureley)**.** *Any property Q that holds* a.a.s. *for $G^\star(n, r)$ also holds* a.a.s. *for $G(n, r)$.*

Proof:

Figure 4.3: An instance of $G(100, 3)$.

$$\begin{aligned} \mathbb{P}\{G \text{ does not have } Q\} &= \mathbb{P}\{G^\star \text{ does not have } Q | G^\star \text{ is simple}\} \\ &= \frac{\mathbb{P}\{G^\star \text{ does not have } Q, G^\star \text{ is simple}\}}{\mathbb{P}\{G^\star \text{ is simple}\}} \\ &\leq \frac{\mathbb{P}\{G^\star \text{ does not have } Q\}}{\mathbb{P}\{G^\star \text{ is simple}\}} \to 0. \end{aligned} \tag{4.9}$$

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This theorem allows us to try to prove properties of interest of the model $G^\star$; the model G then has the same property. Clearly, the converse is not true; e.g., G has no cycles of size 1 and 2, but $G^\star$ does.

4.6 Connectivity of $G(n, r)$

A random regular graph $G(n, r)$ is connected *a.a.s.* for $r \geq 3$. While this might seem surprising at first when we compare with $G(n, p)$, where connectivity required about $\log n$ average vertex degree, recall that the main challenge there was to eliminate isolated vertices; the majority of vertices have already been connected at a much lower average vertex degree of $c = np > 1$. In this sense, we should not be surprised that a constant r is enough to ensure connectivity in $G(n, r)$, as isolated vertices are a priori impossible in this model. We will in fact show a much stronger result, which is that $G(n, r)$ is *a.a.s.* k-connected, which means that there are (at least) k vertex-disjoint paths connecting any pair of vertices. We note in passing that $G(n, 2)$ is not connected *a.a.s.* , and consists of a collection of cycles, and $G(n, 1)$ is simply a random matching.

Theorem 4.5 (Connectivity of $G(n, r)$)**.** *For $r \geq 3$, $G(n, r)$ is r-connected* a.a.s. .

Proof:

We partition the set of vertices into three sets A, S, and B. If there are no edges between A and B, we say that S *separates* A and B. The graph is r-connected if and only if the smallest set S that separates the graph is of size at least r. We denote by T the subset of vertices of S adjacent

to a vertex in A. Let H be the subgraph spanned by $A \cup T$, and let $s = |S|$ and $t = |T|$. We will argue that for every A, $s \geq t \geq r$.

Small component. Fix an arbitrarily large natural number a_0. We first consider a small component A, i.e., of fixed size $a = |A| < a_0$. For $a = 1$, the assertion is immediate. For $a = 2$, $A = \{u, v\}$, we distinguish two cases. If there is no edge (u, v), there are r edges incident to u that need to go to distinct vertices in T. If there is an edge (u, v), then there can be at most one vertex in T adjacent to both u and v, as otherwise $v(H) < e(H)$ (which implies that H does not appear *a.a.s.* , c.f. the proof of Theorem 4.2). So for both cases, $t \geq r$.

For $a > 2$, we lower-bound t and therefore s. The subgraph H contains $a + t$ vertices and at least $(ra + t)/2$ edges, because there are by definition ar stubs in A and t stubs in T in the spanned subgraph H (recall that every vertex in T has at least one edge into A).

Therefore, the condition $v(H) \geq e(H)$ becomes

$$s \geq t \geq a(r - 2). \tag{4.10}$$

This shows that $s \geq r$ *a.a.s.* for a fixed $a \geq 3$ and $r \geq 3$, and therefore this holds over all $3 \leq a \leq a_0$ by a union bound.

Large component. We have shown the above result only for fixed a. The proof for large $a > a_0$ is left as an exercise. The only difficulty is that the number of terms grows with n.

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In fact, it is possible to show much more than that: that the size of the separating set is in fact much larger than r for large a.

It is also worth pointing out that many of the results in this chapter allow for a degree $r = r(n)$ that grows slowly with n.

4.7 General degree distributions

We close this chapter with a brief discussion of an elegant result that generalizes our study of the emergence of the giant component in the models $G(n, p)$ and $G(n, r)$. In this model, the empirical degree distribution $\lambda = \{\lambda_i\}$ is given a priori. A graph from $G(n, \lambda)$ has $n\lambda_i$ vertices of degree i. Clearly, $G(n, p) = G(n, \text{Binom}(n, p))$, and $G(n, r) = G(n, \{\lambda_r = 1\})$.

The model to sample from $G(n, \lambda)$ generalizes the matching model we studied above for $G(n, r)$: we generate stubs for each vertex, then randomly connect the stubs to form a matching, then project and condition on the graph being simple. What is different is that we generate classes of vertices with different degrees, to match the empirical distribution λ (i.e., we have roughly $n\lambda_i$ vertices of degree i, and $ni\lambda_i$ stubs adjacent to a degree i vertex).

Molloy and Reed [3] show the following simple criterion for the emergence of a giant component, subject to some technical conditions that we do not discuss here. Define $Q(\lambda) = \sum_{i \geq 1} i(i - 2)\lambda_i$. If $Q(\lambda) < 0$, then $G(n, \lambda)$ has only small components; if $Q(\lambda) > 0$, then it does have a giant component.

While proving this result is quite involved, we can easily develop an intuition of why the function $Q(\lambda)$ determines the appearance of a giant component. We consider for this the same component discovery process as in the discussion of $G(n, p)$, where A_k is the set of active vertices. We can then view $Q(\lambda)$ as the expected difference between A_{k+1} and A_k. Suppose at the ith step, we saturate a vertex of degree i; this means that we remove this vertex from the active set; there are $i - 1$ neighbors of this vertex that are added, for a total change of $(i - 2)$.

What is the probability of hitting a vertex of degree i? For this, it is important to note that this probability is not proportional to λ_i, but rather to $i\lambda_i$. This is because we sample edges, rather than

vertices, which gives a bias towards higher-degree vertices. The function $Q(\lambda)$ is therefore proportional to the expected change in the active set; if this change is positive, then the discovery process is likely to either die early, or to give rise to a giant component, in analogy to the proofs in Section 3.4.