Networks out of Control:

Homework Set 3

Exercise 1

Stirling's formula ($n! \sim n^n e^{-n}\sqrt{2\pi n}$) is very accurate approximation of $n!$, which is very helpful to obtain asymptotic scaling laws of different properties in large networks. For a random regular graph $G(n,r)$, we use a random matching model $G^*(n,r)$. We have seen that there are $(nr-1)!! = (nr-1)(nr-3)\dots 3$ distinct matching, given that nr is even. We want to find a similar approximation of $(n-1)!!$, given that n is even. Using Stirling's formula (after some manipulations on $(n-1)!!$), show that

$$(n-1)!! \sim \alpha n^{n/2} e^{-n/2}.$$

for some constant $\alpha > 0$ that you need to find.

Exercise 2

Complete the proof of the a.a.s. r-connectivity of $G(n,r)$ when $r \geq 3$, for the case of large $a = |A|$ ($a > a_0$ where a_0 is an arbitrarily fixed large natural number), where the set of vertices is partitioned into three sets A, S, and B, so that S separates A and B.

Hint: The graph is r-connected if and only if the smallest set S that separates the graph is of size at least r. Denote by T the subset of vertices of S adjacent to a vertex in A, and let H be the subgraph spanned by $A \cup T$. Let $s = |S|$ and $t = |T|$. Prove the theorem by contradiction, and assume thus that $t < r$. Observe that you can always choose a_0 large enough so that the expected number of subgraphs H is less than n^{-2}. Show then that the probability of finding one such subgraph H is also less than n^{-2}, and conclude with a union bound on all possible values of $a > a_0$).

Exercise 3

We compare the sampling process of a graph from the $G(n,p)$ model, conditioned on being r-regular, with sampling it from the $G(n,r)$ model. Assume as usual that nr is even.

1. Let us sample a graph from $G(n,p)$, the distribution of random graphs with vertex label set $[n]$ where each pair of vertices has an edge independently with probability p, conditioned on the graph being r-regular. Is this equivalent to sampling a graph uniformly from $\mathcal{G}(n,r)$, the set of labeled (simple) graphs with vertex label set $[n]$ and constant degree r?

More precisely, let X be generated from the $G(n,p)$ model and Y be generated from the $G(n,r)$ model, i.e., sampled uniformly at random from the set of r-regular graphs $\mathcal{G}(n,r)$. If G is a r-regular graph, is $\mathbb{P}\{X = G | X \text{ is } r\text{-regular}\}$ the same as $\mathbb{P}\{Y = G\}$?

Hint: compute the probability of sampling a graph from the $G(n,p)$ model with a number $l = nr/2$ of edges.

2. If your answer to the previous question is negative, give an example of a realization $G \in \mathcal{G}(n, r)$ for which $\mathbb{P}\{X = G \mid X \text{ is } r\text{-regular}\} \neq \mathbb{P}\{Y = G)\}$. If your answer is positive, then why do we not use this model to prove properties of $G(n, r)$?