8

Lattice Bond Percolation

8.1 Lattice bond model; percolation probability

In this section, we formalize the bond model that will be the basis of the chapter, which follows Grimmett's textbook citeGrimmett1999; see also the definitions given in Chapter 2. We consider the d-dimensional lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, where the set of edges $\mathbb{E}^d$ connects sites $(x, y) = ((x_1, \ldots, x_d), (y_1, \ldots, y_d))$ located at the vertices of $\mathbb{Z}^d$ for which the Manahattan distance, defined by

$$\delta(x, y) = \sum_{i=1}^{d} |x_i - y_i|$$

is no more than one: $\delta(x, y) \le 1$. The edges of $\mathbb{E}^d$ connect thus adjacent vertices of $\mathbb{Z}^d$.

Let $0 \le p \le 1$. We declare an edge of $\mathbb{E}^d$ to be *open* with probability p, and closed otherwise, independently of all other edges.

We denote by $C(x)$ the part of $\mathbb{L}^d$ containing the set of vertices connected by open paths to vertex x and the open edges of $\mathbb{E}^d$ connecting such vertices. By translation invariance of the lattice and the probability measure $\mathbb{P}_p$, the distribution of $C(x)$ does not depend on the vertex x. We therefore take in general $x = 0$, and denote by C the open cluster at the origin: $C = C(0)$.

If A and B are sets of vertices of $\mathbb{L}^d$, we write $A \leftrightarrow B$ to express the fact that there exists an open path connecting some vertex of A to some vertex of B. For example, $C(x) = \{y \in \mathbb{Z}^d \mid x \leftrightarrow y\}$. We write ∂A to denote the surface of A, which is the set of vertices of A which are adjacent to some vertex that does not belong to A. A typical subset of vertices is a *box*, defined as

$$B(n) = [-n, n]^d = \left\{x \in \mathbb{Z}^d \mid \max_{1 \le i \le d} \{|x_i|\} \le n\right\}$$

for some $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We write $B(n, x)$ for the box $x + B(n)$ having side-length $2n$ and center at x. We will also work often with "diamond" boxes

$$S(n) = \{x \in \mathbb{Z}^d \mid \delta(0, x) \le n\}$$

or more general rectangular boxes. We also write $S(n, x)$ for the diamond box $x + S(n)$ centered in x.

8.2 Percolation in the 2-dim Lattice

8.2.1 Percolation Probability and Existence of a Phase Transition

The main quantity of interest in percolation theory is the probability that the origin belongs to a cluster with an infinite number of vertices, given by (2.1), which we recall here:

$$\theta(p) = \mathbb{P}_p(|C| = \infty). \tag{8.1}$$

By space invariance, $\theta(p)$ is the probability that any node belongs to an infinite cluster.

Define the *critical* (or *percolation*) *threshold* by (2.2), which reads

$$p_c = \sup \{p \mid \theta(p) = 0\}. \tag{8.2}$$

We compute the exact value of p_c and even $\theta(p)$ for the regular (binary) tree in Chapter 2. For the lattice $\mathbb{L}^d$, we will not be able to compute analytically $\theta(p)$. Computing the exact value of p_c is already a challenge, and still remains an open problem for dimensions larger than 2. In this chapter, we will compute it for dimension $d = 2$, where Kesten closed the conjecture after more than two decades of research. We can already frame here the value of p_c within 1/3 and 2/3 thanks to the following theorem.

Theorem 8.1 (Non trivial phase transition)**.** *The percolation threshold in $\mathbb{L}^2$ is such that $1/3 \leq p_c \leq 2/3$.*

The proof makes use of a technique that will prove to be quite powerful in $d = 2$ dimensions, but that does not generalize well to higher dimensions, which is to work with the *planar dual* graph. If G is a planar graph, drawn in the plane in such a way that edges intersect only at their common vertices, then the dual graph G_d is obtained by putting a vertex in every face of G, and by joining two such vertices by an edge whenever the corresponding faces of G share a common edge. When $G = \mathbb{L}^2$, its dual $G_d = \mathbb{L}^2_d$ is isomorphic to $\mathbb{L}$. The vertices of the dual lattice $\mathbb{L}^2_d$ are placed at the centers of the squares of $\mathbb{L}^2$, i.e. are the set $\{(i + 1/2, j + 1/2) \mid (i, j) \in \mathbb{Z}^2\}$, and its edges connect adjacent vertices. To every edge of $\mathbb{L}^2$ corresponds exactly one edge of L^2_d, and vice-versa. We declare an edge of the dual lattice L^2_d to be open (resp., close) if and only if its corresponding edge in the lattice $\mathbb{L}^2$ is open (resp., close), as shown in Figure 8.1. This results in a bond percolation process on the dual lattice with the same open edge probability p.

Proof:

(i) We first prove that $p_c \geq 1/3$. Let $\sigma(n)$ be the number of distinct, loop free paths ("self-avoiding walks") of $\mathbb{L}^d$ having length n and beginning at the origin. The exact value of $\sigma(n)$ is very difficult to compute for already moderate values of n, but an upper bound on $\sigma(n)$ is $4 \cdot 3^{n-1}$. Indeed, walking from the origin, we have first 4 possible edges to take, and then, at each step, up to 3 different edges. Let $N(n)$ be the number of such paths that are open. Since each path is open with probability p^n,

$$\mathbb{E}_p[N(n)] = \sum_{s=1}^{\sigma(n)} \mathbb{E}_p\left[1_{\{\text{ path } s \text{ is open}\}}\right] = \sigma(n)p^n.$$

The origin belongs to an infinite open cluster if there are open paths of all possible lengths beginning

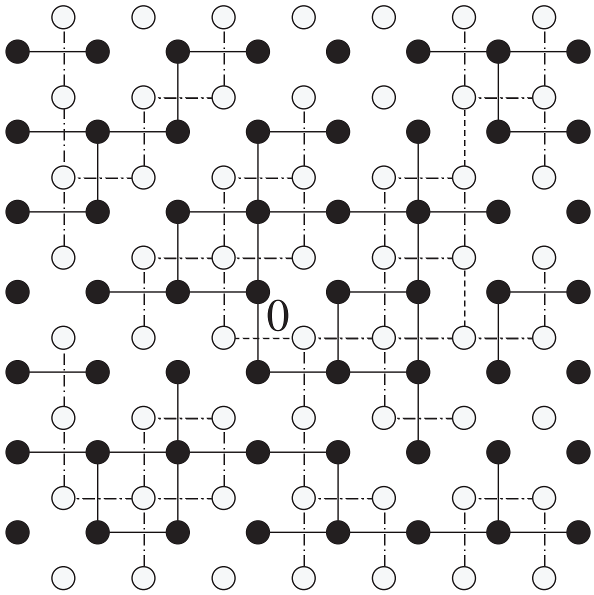


Figure 8.1: A portion of the lattice $\mathbb{L}^2$ (whose vertices are represented by full circles, open edges by plain lines) and its dual (whose vertices are represented by empty circles, and open edges by dashed lines).

at the origin, hence for all $n \in \mathbb{N}^*$

$$\begin{aligned}\theta(p) &\leq \mathbb{P}_p(N(n) \geq 1) = \sum_{s=1}^{\sigma(n)} \mathbb{P}_p(N(n) = s) \\ &\leq \sum_{s=1}^{\sigma(n)} s\mathbb{P}_p(N(n) = s) = \mathbb{E}_p[N(n)] = \sigma(n)p^n \\ &\leq \frac{4}{3}(3p)^n.\end{aligned}$$

Letting $n \to \infty$, we find that $\theta(p) = 0$ if $p < 1/3$. Hence $p_c \geq 1/3$.

(ii) We next prove $p_c \leq 2/3$. Let $m \in \mathbb{N}^*$, and let F_m be the event that there exists a closed circuit in the dual lattice $\mathbb{L}_d^2$ containing the box $B(m) = [-m, m] \times [-m, m]$ in its interior, and let G_m be the event that all edges of $B(m)$ are open. The origin belongs to an infinite cluster if F_m does not occur and G_m does occur, see Figure 8.2. Since these events are defined on disjoint sets of edges, they are independent and we have therefore that

$$\theta(p) \geq \mathbb{P}_p(\overline{F}_m \cap G_m) = \mathbb{P}_p(\overline{F}_m)\mathbb{P}_p(G_m). \tag{8.3}$$

Now, $\mathbb{P}_p(G_m) > 0$ and so all we need to do is to show that $\mathbb{P}_p(\overline{F}_m) > 0$ for $p \geq 2/3$.

Let $\gamma(n)$ be the number of self-avoiding circuits in the dual lattice L_d^2 surrounding the origin and of length n, and which consists of a single loop (In other words, the degree of every vertex of such a closed circuit is 2: we will speak of a "self-avoiding circuit"). Each such circuit must pass through a vertex of the form $(i + 1/2, 1/2)$ for some $0 \leq i \leq n - 1$, because (a) to surround the origin, it has to pass through a vertex $(i + 1/2, 1/2)$ for some $i \geq 0$, and (b) it cannot pass through a vertex $(i + 1/2, 1/2)$ for some $i \geq n$ since it would then be at least $2n$. Such a circuit contains a self-avoiding walk of length $n - 1$ starting from one of the n vertices $(i + 1/2, 1/2)$ for some

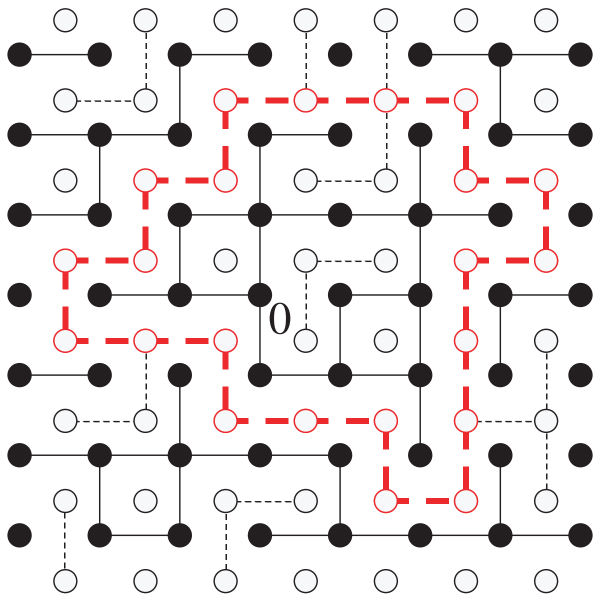


Figure 8.2: A portion of the lattice $\mathbb{L}^2$ (whose vertices are represented by full circles, open edges by plain lines) and its dual (whose vertices are represented by empty circles, and closed edges by dashed lines) Observe that there is a circuit of closed dual edges surrounding the origin (set in red bold on the figure), which therefore belongs to a finite open cluster.

$0 \leq i \leq n-1$. Therefore

$$\gamma(n) \leq n\sigma(n-1).$$

Now, the occurrence of the event F_m requires that there is at least one such closed circuit, with a length of at least $8m$ hops to contain $B(m)$:

$$\begin{aligned} F_m &\subseteq \{\text{there is at least one closed circuit of length } 8m \text{ surrounding } 0\} \\ &= \bigcup_{\text{circuit } g \text{ of length at least } 8m} \{g \text{ is closed}\}. \end{aligned}$$

Using the union bound, we get therefore that

$$\begin{aligned} \mathbb{P}_p(F_m) &\leq \sum_{\text{circuit } g \text{ of length at least } 8m} \mathbb{P}_p\left(g \text{ is closed}\right) \\ &= \sum_{n=8m}^{\infty} \sum_{\text{circuit } g \text{ of length } n} \mathbb{P}_p\left(g \text{ is closed}\right) \\ &\leq \sum_{n=8m}^{\infty} \gamma(n)\left(1-p\right)^n \\ &\leq \frac{4(1-p)}{3} \sum_{n=8m}^{\infty} n\left(3(1-p)\right)^{n-1}. \end{aligned} \tag{8.4}$$

If $p > 2/3$, this sum converges to some finite value, and we take m large enough so that it is less than $1/2$. Consequently, from (8.3), we get

$$\theta(p) \geq \mathbb{P}_p(\overline{F}_m)\mathbb{P}_p(G_m) \geq P_p(G_m)/2 > 0,$$

which proves the result.

■

For the 2-dim bond model, the exact value of p_c is known, and we will compute it in a few chapters, as it requires quite a lot of work. The simulations of Figure 8.3 show a 40×40 lattice. Although non infinite, the phase transition is already visible: all clusters are finite for $p = 0.3$ and $p = 0.49$, whereas one giant cluster is present for $p = 0.51$ and clearly for $p = 0.7$.

Figure 8.4 displays an estimate of the percolation probability $\theta(p)$, for a 5000×5000 lattice. Despite the finite size of the lattice, the phase transition, which stricto sensu only occurs for an infinite lattice, appears quite clearly on the figure.

In higher dimensions, the d-dim lattice $\mathbb{L}^d$ can always be embedded in a $(d+1)$-dim lattice $\mathbb{L}^{d+1}$, and therefore if the origin belongs to an infinite cluster in $\mathbb{L}^d$, it also belongs to an infinite cluster in $\mathbb{L}^{d+1}$. Therefore, the percolation threshold is a decreasing function of d: $p_c(d+1) \leq p_c(d)$.

A direct corollary of Theorem 8.1 is that the probability that there exists an infinite open cluster, which we denote by $\hat{\theta}(p)$ follows a zero-one law (see Appendix 8.7).

Corollary 8.1. *Existence of an open cluster The probability that there exists an open infinite cluster is*

$$\hat{\theta}(p) = \begin{cases} 0 & \text{if} \quad p < p_c \\ 1 & \text{if} \quad p > p_c. \end{cases}$$

We will however give a stronger result in the chapter on the super-critical phase.

8.2.2 Mean Cluster Size

The *mean size of an open cluster* (2.5) $\chi(p) = \mathbb{E}_p[|C|]$, which by translation invariance is the expected number of vertices in the open cluster at the origin, can be expressed as

$$\chi(p) = \mathbb{E}_p[|C|] = \sum_{n=1}^{\infty} n\mathbb{P}_p(|C| = n) + \infty\mathbb{P}_p(|C| = \infty).$$

If $p > p_c$, then we see that $\chi(p) = \infty$. The converse is not obvious, and it will require quite some work to prove in the next chapter that if $p < p_c$ then $\chi(p) < \infty$. Figure 8.5 displays an estimate of the mean cluster size $\chi(p)$, for the 5000×5000 lattice.

In the supercritical phase, since the mean cluster size is infinite, one is more interested in the mean size of the finite clusters, which we denote $\chi^f(p)$ and which is defined as the mean of $|C|$ on the event that $|C|$ is finite:

$$\chi^f(p) = \mathbb{E}_p[|C|; |C| < \infty] = \mathbb{E}_p\left[|C|\, 1_{\{|C|<\infty\}}\right] = \mathbb{E}_p[|C| \mid |C| < \infty](1 - \theta(p)). \tag{8.5}$$

8.3 Three Inequalities for Increasing Events

This section introduces three technical devices, which will be repeatedly used in the proofs of theorems in the following sections. We need first the following definition.

Definition 8.1 (Increasing event)**.** *A random variable X is* increasing *on $(\Omega, \mathcal{F})$ if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. It is decreasing if $-X$ is increasing. An event $A \in \mathcal{F}$ is increasing whenever its indicator function is an increasing variable, i.e. if $1_A(\omega) \leq 1_A(\omega')$ whenever $\omega \leq \omega'$.*

It is easy to show that if A is an increasing event, then $\mathbb{P}_p(A) \leq \mathbb{P}_{p'}(A)$ whenever $p \leq p'$.

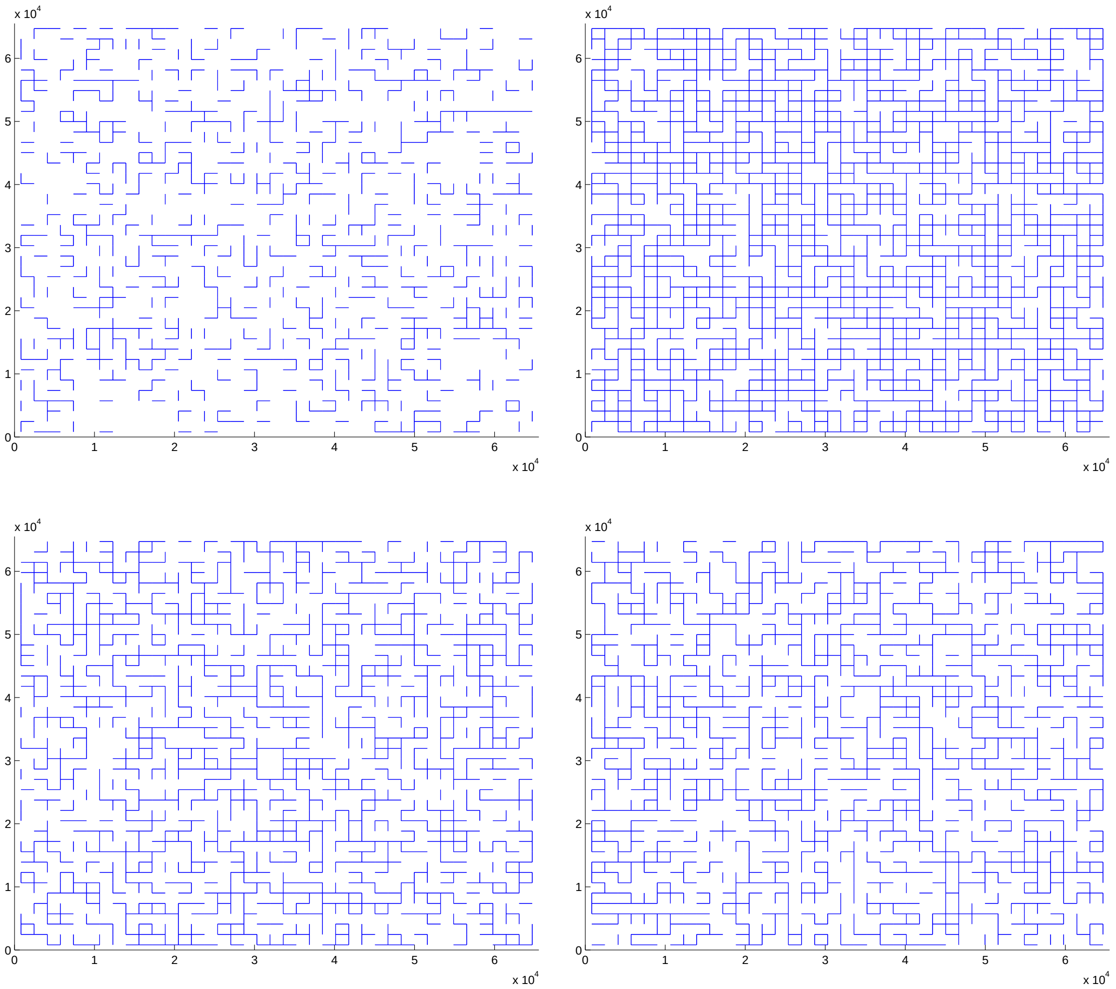


Figure 8.3: A simulation of bond percolation in a 40×40 lattice for different values of the open edge probability: $p = 0.3$ (upper left), $p = 0.49$ (bottom left), $p = 0.51$ (bottom right) and $p = 0.7$ (top right). Only the open edges are shown. A careful inspection of the two graphs at bottom reveals the emergence of a giant open cluster for $p \geq 0.51$, which was absent when $p \leq 0.49$.

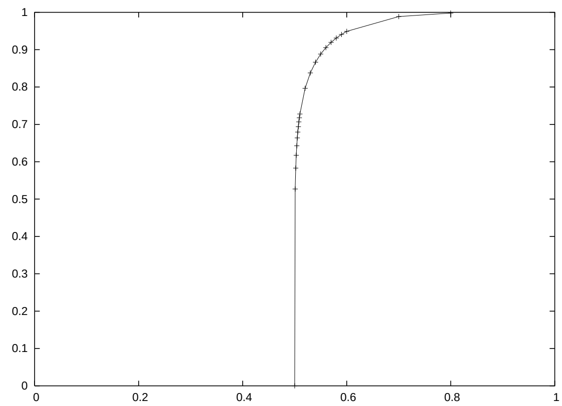


Figure 8.4: An estimation of the percolation probability $\theta(p)$, for a 5000×5000 lattice, as a function of p.

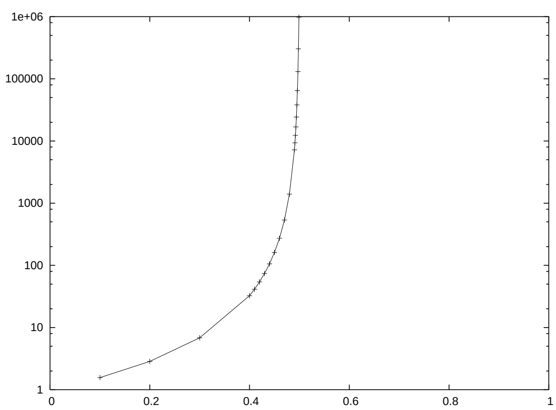


Figure 8.5: An estimation of the mean cluster size $\chi(p)$, for the 5000×5000 lattice, as a function of p.

8.3.1 FKG inequality

The FKG inequality (named after Fortuin Kasteleyn and Ginibre) was first shown by Harris in 1960. It expresses the fact that increasing events can only be positively correlated.

Theorem 8.2 (FKG inequality)**.** *If A and B are two increasing events, then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

We establish the FKG inequality in the case where A and B are depend on finitely many edges. The proof of the FKG inequality when A and/or B depend on infinitely many edges is found in [7]. The FKG inequality also holds when both A and B are two decreasing events.

Proof:

Let $X = 1_A$ and $Y = 1_B$ be the indicators of the increasing events A and B, which are increasing random variables. We can then reformulate the FKG inequality as $\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\mathbb{E}_p[Y]$. Suppose that X and Y depend only on the state of edges $e_1, e_2, \ldots, e_n$ for some integer n. We prove the FKG inequality by induction.

Suppose first that $n = 1$, so that X and Y are only functions of the state $\omega(e_1)$ of the edge e_1. Pick any two states $\omega_1, \omega_2 \in \{0, 1\}$. Since both X and Y are increasing random variables,

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$$

with equality if $\omega_1 = \omega_2$. Therefore

$$\begin{aligned}
0 &\leq \sum_{\omega_1=0}^{1}\sum_{\omega_2=0}^{1}(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2))\mathbb{P}_p(\omega(e_1) = \omega_1)\mathbb{P}_p(\omega(e_1) = \omega_2) \\
&= \sum_{\omega_1=0}^{1} X(\omega_1)Y(\omega_1)\mathbb{P}_p(\omega(e_1) = \omega_1) + \sum_{\omega_2=0}^{1} X(\omega_2)Y(\omega_2)\mathbb{P}_p(\omega(e_1) = \omega_2) \\
&\quad - \sum_{\omega_1=0}^{1}\sum_{\omega_2=0}^{1}(X(\omega_1)Y(\omega_2) + X(\omega_2)Y(\omega_1)\mathbb{P}_p(\omega(e_1) = \omega_1)\mathbb{P}_p(\omega(e_1) = \omega_2) \\
&= 2\left(\mathbb{E}_p[XY] - \mathbb{E}_p[X]\mathbb{E}_p[Y]\right).
\end{aligned}$$

Let $1 < k \leq n$. Suppose now that the claim holds for all $m < k$, and that X and Y depend only on the states $\omega(e_1), \ldots, \omega(e_k)$ of the edges $e_1, \ldots, e_k$. Then, given $\omega(e_1), \ldots, \omega(e_{k-1})$, X and Y only depend on the state $\omega(e_k)$ of the edge e_k, and proceeding as above, we have that

$$\mathbb{E}_p[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})] \geq \mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]$$

and thus

$$\begin{aligned}
\mathbb{E}_p[XY] &= \mathbb{E}_p\left[\mathbb{E}_p[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right] \\
&\geq \mathbb{E}_p\left[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right].
\end{aligned}$$

Now, $\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]$ and $\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]$ are increasing functions of the state of the $(k-1)$ edges $e_1, \ldots, e_{k-1}$. By induction, it implies that

$$\begin{aligned}
\mathbb{E}_p[XY] &\geq \mathbb{E}_p\left[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right] \cdot \mathbb{E}_p\left[\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right] \\
&= \mathbb{E}_p[X]\mathbb{E}_p[Y].
\end{aligned}$$

■

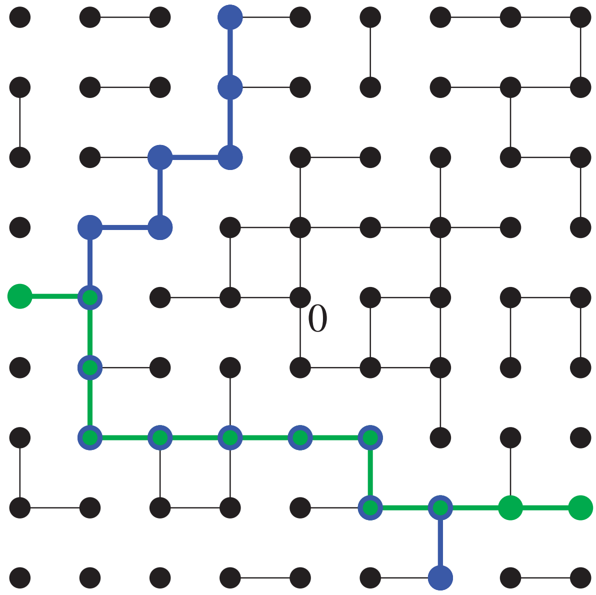


Figure 8.6: The box $B(5)$ with a LR and a TB open crossing.

As an example of application of the FKG inequality, consider the 2-dim. box $B(n)$, and let A be the event that there is an open path joining a vertex of the top face of $B(n)$ to the bottom face of $B(n)$ (we call such a path a TB (top-bottom) (open) crossing of $B(n)$, and B be the event that there is an open path joining a vertex of the left face of $B(n)$ to the right face of $B(n)$ (we call such a path a LR (left-right) (open) crossing of $B(n)$, as shown in Figure 8.6. Then the probability that there are both a TB and LR open crossings of $B(n)$ is at least the product of the probabilities that there is a TB open crossing and that there is a LR open crossing.

8.3.2 BK inequality

The BK inequality (named after van den Berg and Kesten, who proved it in 1985) can be regarded as the reverse of the FKG inequality, with one difference: it applies to the event $A \circ B$ that two increasing events A and B occur on *disjoint* sets of edges, and not to the larger event $A \cap B$ that events A and B occur on any sets of edges. $A \circ B$ is the set of configurations $\omega \in \Omega$ for which there are disjoint sets of open edges such that the first set guarantees the occurrence of A and the second set guarantees the occurrence of B. The formal definition is as follows.

Definition 8.2 (Disjoint occurrence). *Let A and B be two increasing events which depends on the states $\omega(e_1), \ldots, \omega(e_n)$ of n distinct edges $e_1, \ldots e_n$ of $\mathbb{L}^d$. Each such configuration is specified uniquely by the subset $K(\omega) = \{e_i \mid \omega(e_i) = 1\}$ of open edges among these n edges. Then $A \circ B$ is the set of ω for which there exists a subset $H \subset K(\omega)$ such that any ω' determined by $K(\omega') = H$ is in A and any ω'' determined by $K(\omega'') = K(\omega) \setminus H$ is in B.*

Theorem 8.3 (BK inequality). *If A and B are two increasing events, then*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

We only sketch the intuition behind the proof of van den Berg when A and B are the existence of two open paths between different sets of vertices. The full proof is given in [7]. The BK inequality also holds when both A and B are two decreasing events.

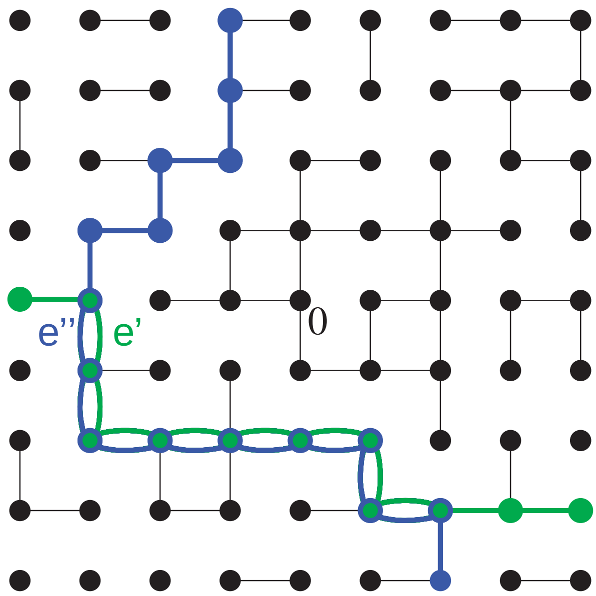


Figure 8.7: Construction of two independent copies of the lattice

Proof:

(Sketch) Let G be a finite subgraph of $\mathbb{L}^d$. Let A (respectively, B) be the event that there exists an open path between vertices u and v (respectively, x and y). $A \circ B$ is the event that there exist two disjoint open paths from u to v and from x to y. Let e be an edge of G. Replace e by two parallel edges e' and e'', having the same end vertices, each of which being open with the same probability p, independently of each other and of all other edges. The splitting of edge e in the two edges e' and e'' can only make our search for two disjoint open paths easier: indeed, if in graph G two paths from u to v and from x to y had to use the same edge e, they now can replace this common edge by the two distinct edges e' and e''. The probability of finding two disjoint open paths from from u to v and from x to y can therefore only increase or remain equal after this splitting. We continue this splitting process, as shown in Figure 8.7, replacing every edge f of G by two parallel edges f' and f''. At each stage, we look for two open paths, the first one avoiding all edges marked $''$ and the second one all edges marked $'$. The probability of finding two such paths can only increase or remain equal at each stage. When all edges of G have been split in two, we end up with two independent copies of G, in the first of which we look for an open path connecting u to v, and in the second of which we look for an open path connecting x to y. Since such paths occur independently in each copy of G, the probability that they both occur is $\mathbb{P}_p(A)\mathbb{P}_p(B)$.

■

As an example of application of the BK inequality, consider again the 2-dim. box $B(n)$, and let A be the event that there is an open TB crossing path of $B(n)$ to the bottom face of $B(n)$, and B be the event that there is an open LR crossing, which is edge-disjoint with A. This event does not occur in the example of Figure 8.6, but does occur for the example of Figure 8.8. Then the probability that there are edge disjoint TB and LR open crossings of $B(n)$ is no more than the product of the probabilities that there is a TB open crossing and that there is a LR open crossing.

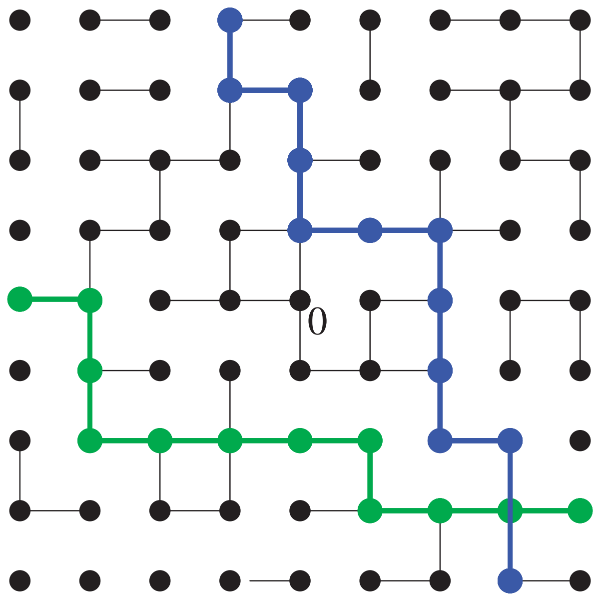


Figure 8.8: The box $B(5)$ with edge-disjoint LR and TB open crossings.

8.3.3 Multiplicity of edge-disjoint paths

A last result is useful to relate the probability that r edge-disjoint paths cross a given portion of the lattice to the probability that at least one such path exists. It follows directly from Theorem 2.45 in [7] and the remarks thereafter.

Lemma 8.1. *Let A_n be the event that there exists an open path between the left and right sides of $B(n)$ and $I_r(A_n)$ the event that there exist r edge-disjoint such LR crossings. We have*

$$1 - \mathbb{P}_p(I_r(A_n)) \leq \left(\frac{p}{p-p'}\right)^r [1 - \mathbb{P}_{p'}(A_n)]$$

for any $0 \leq p' < p \leq 1$.

Proof:

Let $\{X(e), e \in \mathbb{E}^2\}$ be a collection of i.i.d. random variables indexed by the edge set $\mathbb{E}^2$, uniformly distributed on $[0,1]$. Let η_p be the configuration of edges defined by

$$\eta_p(e) = \begin{cases} 1 & \text{if} \quad X(e) < p \\ 0 & \text{if} \quad X(e) \geq p \end{cases}$$

for some $0 \leq p \leq 1$ and all $e \in \mathbb{E}^d$. Observe that

$$\begin{aligned} \mathbb{P}(\eta_p(e) = 0) &= \mathbb{P}(X(e) \geq p) = 1 - p \\ \mathbb{P}(\eta_p(e) = 1) &= \mathbb{P}(X(e) < p) = p. \end{aligned}$$

Hence $\mathbb{P}_p(A_n) = \mathbb{P}(\eta_p \in A_n)$.

Observe that the configuration of edges η_p does not have r LR edge-disjoint crossings of $B(n)$ (in other words, $\eta_p \notin I_r(A_n)$), if and only if there is a (possibly empty) collection C of edges, with (i) $|C| \leq r$, (ii) $\eta_p(e) = 1$ for all $e \in C$, and (iii) the configuration $(\eta_p \setminus C)$ obtained by declaring all edges in C closed does not have a LR open path that crosses $B(n)$, i.e. $(\eta_p \setminus C) \notin A_n$. Indeed, there are less than r edge-disjoints LR crossings if and only if we can find at most r edges that form the minimal cutset of the graph between the left and right sides of $B(n)$.

Suppose that $\eta_p \notin I_r(A_n)$. Then

$$\begin{aligned}
\mathbb{P}(\eta_{p'} \notin A_n \mid \eta_p \notin I_r(A_n)) &= \mathbb{P}(\eta_{p'} \notin A_n \mid \text{there is a set } C \text{ verifying (i) - (iii) above}) \\
&= \mathbb{P}(\eta_{p'}(e) = 0 \text{ for all } e \in C \mid \text{there is a set } C \text{ verifying (i) - (iii) above}) \\
&= \frac{\mathbb{P}(\{\text{there is a set } C \text{ verifying (i) - (iii) above}\} \cap \{\eta_{p'}(e) = 0 \text{ for all } e \in C\})}{\mathbb{P}(\text{there is a set } C \text{ verifying (i) - (iii) above})} \\
&= \frac{\mathbb{P}(\{\text{there is a set } C \text{ verifying (i) - (iii) above}\} \cap \{p' \leq X(e) < p \text{ for all } e \in C\})}{\mathbb{P}(\{\text{there is a set } C \text{ verifying (i) - (iii) above}\} \cap \{X(e) < p \text{ for all } e \in C\})} \\
&\geq \left(\frac{p-p'}{p}\right)^r
\end{aligned}$$

and

$$\mathbb{P}(\{\eta_{p'} \notin A_n\} \cap \{\eta_p \notin I_r(A_n)\}) \geq \left(\frac{p-p'}{p}\right)^r \mathbb{P}(\eta_p \notin I_r(A_n)),$$

from which we deduce that

$$\begin{aligned}
1 - \mathbb{P}_p(I_r(A_n)) &= \mathbb{P}_p(\eta_p \notin I_r(A_n)) \\
&\leq \left(\frac{p}{p-p'}\right)^r \mathbb{P}(\{\eta_{p'} \notin A_n\} \cap \{\eta_p \notin I_r(A_n)\}) \\
&\leq \left(\frac{p}{p-p'}\right)^r \mathbb{P}(\eta_{p'} \notin A_n) \\
&= \left(\frac{p}{p-p'}\right)^r [1 - \mathbb{P}_{p'}(A_n)].
\end{aligned}$$

■

This theorem is in fact much more general. First, it is not restricted to a portion of $\mathbb{L}^d$, which is a box $B(n)$. Second, it applies to any increasing event A, if we define $I_r(A)$ to be the *interior* of A with depth r, defined as the set of configuration in A, which remain in A even if the states of up to r edges is modified (see [7]).

8.4 Subcritical phase

In this section, we study the situation in the subcritical phase, when $p < p_c$ and $d \geq 2$. In this case, we know that the open cluster C containing the origin is almost surely finite since $\theta(p) = \mathbb{P}_p(|C| = \infty) = 0$.

The main result from this section is that the radius of the mean cluster size decreases exponentially when $p < p_c$. As a result, the mean cluster size is finite in the subcritical phase: $\chi(p) < \infty$ when $p < p_c$.

8.4.1 Exponential decrease of the radius of the mean cluster size

Let $S(n)$ be the diamond of radius n (i.e., the ball of radius n with the Manhattan distance), that is the set of all vertices $x \in \mathbb{Z}^d$ for which $\delta(0, x) = |x| \leq n$. Let $A_n = \{0 \leftrightarrow \partial S(n)\}$ be the event that there exists an open path connecting the origin to any vertex lying on the surface of $S(n)$, which we denote by $\partial S(n)$. Defining the radius of C by $\mathrm{rad}(C) = \max_{x \in C}\{|x|\}$, we see that $A_n = \{\mathrm{rad}(C) \geq n\}$.

Theorem 8.4 (Exponential decay of the radius of an open cluster)**.** *If $p < p_c$, there exists $\psi(p) > 0$ such that*

$$\mathbb{P}_p(rad(C) \geq n) = \mathbb{P}_p(0 \leftrightarrow \partial S(n)) = \mathbb{P}_p(A_n) < \exp(-n\psi(p)).$$

The proof of this challenging theorem, due to Menshikov (1986) is long, and can be found in [7].

A number of easier results can be deduced from Theorem 8.4, which show that in the subcritical phase, basically all metrics related to the size of the cluster containing the origin are exponentially decreasing with the size.

8.4.2 Connectivity function and correlation length

We begin with the *connectivity function* $\mathbb{P}_p(x \leftrightarrow y)$ which is defined as the probability that two vertices x and y are connected together by an open path. By translation invariance, we can take $y = 0$. Moreover, in the sake of simplicity and without loss of generality, we assume that x is positioned along the x-axis : $x = x_n$ where u_n is the d-dimensional vector $u_n = (n, 0, \ldots 0)$.

Theorem 8.5 (Exponential decay of connectivity function)**.** *If $0 < p < p_c$, there exists $0 < \xi(p) < \infty$ and a constant $\kappa > 0$ independent of p, such that*

$$\kappa p n^{4(1-d)} \exp(-n/\xi(p)) \leq \mathbb{P}_p(0 \leftrightarrow u_n) \leq \exp(-n/\xi(p)). \tag{8.6}$$

This shows that

$$\mathbb{P}_p(0 \leftrightarrow u_n) \approx \exp(-n/\xi(p)).$$

The function $\xi(p)$ is called the *correlation length*, and one can show that $\xi(p) = 1/\psi(p)$, where $\psi(p)$ is the exponent in Theorem 8.4.

We prove only the upper bound, the proof for the lower bound is longer but builds essentially on a similar argument, based on the sub-additive limit theorem, which we recall here.

Lemma 8.2 (Sub-additive limit theorem)**.** *Let $\{x_n, n \in \mathbb{N}^*\}$ be a sub-additive sequence of real non negative numbers, i.e. a sequence of real non negative numbers such that*

$$x_{m+n} \leq x_m + x_n \tag{8.7}$$

for all $m, n \in \mathbb{N}^*$, the limit

$$\overline{x} = \lim_{n\to\infty} x_n/n$$

exists and is finite. Moreover,

$$\overline{x} = \inf_{n \in \mathbb{N}^*} x_n/n$$

and thus $x_m \geq m\overline{x}$ for all $m \in \mathbb{N}^*$.

Proof:

We prove only the upper inequality. The starting point is the observation (see Figure 8.9) that

$$\{0 \leftrightarrow u_{m+n}\} \supseteq \{0 \leftrightarrow u_m\} \cap \{u_m \leftrightarrow u_{m+n}\}$$

and thus, by first using the FKG inequality and next by translation invariance,

$$\mathbb{P}_p(0 \leftrightarrow u_{m+n}) \geq \mathbb{P}_p(0 \leftrightarrow u_m)\mathbb{P}_p(u_m \leftrightarrow u_{m+n}) = \mathbb{P}_p(0 \leftrightarrow u_m)\mathbb{P}_p(0 \leftrightarrow u_n).$$

Letting $x_n = -\ln \mathbb{P}_p(0 \leftrightarrow u_n)$, this inequality becomes (8.7), and therefore, by Lemma 8.2, the limit

$$\xi^{-1}(p) = \lim_{n\to\infty} \left(-\frac{\ln \mathbb{P}_p(0 \leftrightarrow u_n)}{n} \right)$$

exists. Moreover, $x_n \geq n/\xi(p)$ for all $n \in \mathbb{N}^*$, which yields the result.

■

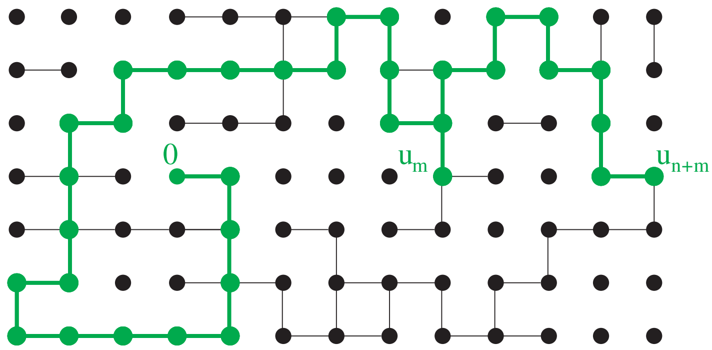


Figure 8.9: The event $\{0 \leftrightarrow u_{m+n}\}$ is more likely than the joint occurrence of events $\{0 \leftrightarrow u_m\}$ and $\{u_m \leftrightarrow u_{m+n}\}$.

8.4.3 Cluster size distribution

It follows from Theorem 8.4 that the distribution number $|C|$ of vertices contained in the open cluster at the origin has an exponentially decreasing tail. A more accurate bound is as follows.

Theorem 8.6 (Exponential decay of the cluster size distribution)**.** *If* $0 < p < p_c$*, there exists* $\lambda(p) > 0$ *such that*

$$\mathbb{P}_p(|C| \geq n) \leq \exp(-n\lambda(p)) \tag{8.8}$$

and there exists $0 < \zeta(p) < \infty$ *such that*

$$\mathbb{P}_p(|C| = n) \leq \frac{(1-p)^2}{p} n \exp(-n\zeta(p)). \tag{8.9}$$

for $n \in \mathbb{N}^*$*.*

One can moreover show that

$$\mathbb{P}_p(|C| = n) \approx \exp(-n\zeta(p)).$$

The theorem is proven in [7].

8.5 Supercritical phase

In this section we study the situation in the supercritical phase, when $p > p_c$ and $d \geq 2$. In this case, we know that there is almost surely an open cluster of infinite size. But how many are there ? We will first prove that there is exactly one such cluster. The next question will be to evaluate the size of the other, finite clusters. We will see in that they decrease sub-exponentially fast. We will prove the result only when $d = 2$, although it holds for $d \geq 3$ as well.

8.5.1 Uniqueness of the infinite open cluster

We follow the approach of Burton and Keane (1989), as exposed in [7], to prove that the infinite open cluster is unique in the supercritical phase.

Theorem 8.7 (Uniqueness of the infinite open cluster)**.** *If* $p > p_c$*, then*

$$\mathbb{P}_p(\textit{there exists exactly one infinite open cluster}) = 1.$$

Proof:

Let Y be the number of infinite open clusters. Because the sample space $\Omega = \prod_{e\in\mathbb{E}}\{0,1\}^e$ is a product space with a space invariant product measure $\mathbb{P}_p$, Y is a translation-invariant function on Ω. A property of translation-invariant functions under ergodic measures is to be almost surely constant. Consequently, there exists some $k \in \mathbb{N}\cup\{\infty\}$ such that $\mathbb{P}_p(Y = k) = 1$.

Since $p > p_c$, $k \neq 0$. We will prove by contradiction that (i) $k \notin [2,\infty[$ and (ii) $k \neq \infty$, which implies therefore that $k = 1$.

(i) Suppose first that $2 \le k < \infty$. As in the previous chapter, denote by $S(n) = \{x \in \mathbb{Z}^d \mid \delta(0,x) = |x| \le n\}$ the diamond of radius n (i.e., the ball of radius n with the Manhattan distance). Let $Y(0)$ be the number of infinite open clusters when all edges of $S(n)$ are closed. As the probability that all edges of $S(n)$ are closed is strictly positive,

$$\mathbb{P}_p(Y(0) = k) = \frac{\mathbb{P}_p(\{Y = k\}\cap\{\text{all edges of } S(n) \text{ are closed}\})}{P_p(\text{all edges of } S(n) \text{ are closed})} = 1.$$

Similarly, if $Y(1)$ denotes the number of infinite open clusters when all edges of $S(n)$ are open, $\mathbb{P}_p(Y(1) = k) = 1$, and therefore $\mathbb{P}_p(Y(0) = Y(1)) = 1$. We always have that $Y(0) \ge Y(1)$, but since there are only a finite number of open infinite clusters, we have $Y(0) = Y(1)$ if and and only if $S(n)$ intersects exactly one such cluster. So, if $M_{S(n)}$ is the number of infinite open clusters intersecting $S(n)$, $\mathbb{P}_p(M_{S(n)} \ge 2) = 0$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have that the diamond $S(n)$ becomes the entire lattice $\mathbb{L}^d$ and therefore that

$$0 = \lim_{n\to\infty} \mathbb{P}_p(M_{S(n)} \ge 2) = \mathbb{P}_p(Y \ge 2), \tag{8.10}$$

a contradiction with $\mathbb{P}(Y = k) = 1$ for some $2 \le k < \infty$.

(ii) Suppose next that $k = \infty$. We use a geometric argument to get a contradiction, which is based on the following object. We call a vertex x a *trifurcation* (see Figure 8.10) if

1. x belongs to an infinite open cluster;
2. there exist exactly three open edges incident to x; and
3. the deletion of x and of the three open edges incident to x splits the infinite open cluster containing x in exactly three disjoint infinite clusters and no finite cluster.

Because of the space invariance of $\mathbb{L}^d$, the probability that a vertex x is a trifurcation is independent of x, and therefore

$$\mathbb{P}_p(x \text{ is a trifurcation}) = \mathbb{P}_p(0 \text{ is a trifurcation}). \tag{8.11}$$

Let us show that this probability is non zero. Let $M_{S(n)}(0)$ be the number of infinite open clusters intersecting $S(n)$ when all edges of $S(n)$ are closed. Clearly, $M_{S(n)}(0) \ge M_{S(n)}$. Therefore

$$\mathbb{P}_p(M_{S(n)}(0) \ge 3) \ge \mathbb{P}_p(M_{S(n)} \ge 3) \to \mathbb{P}_p(Y \ge 3) = 1$$

as $n \to \infty$. Consequently, there is $n \in \mathbb{N}$ such that $\mathbb{P}_p(M_{S(n)}(0) \ge 3) \ge 1/2$, fix n to this value from now on until we have shown the probability of having a trifurcation at the origin is non zero. If $M_{S(n)}(0) \ge 3$, then there exists three vertices $x, y, z \in \partial S(n)$ lying in three distinct infinite open clusters. Moreover, there are three paths inside $S(n)$ joining the origin to respectively x, y, z, such that the origin is the unique vertex common to any two of them, and each touches exactly one vertex on $\partial S(n)$. For a configuration of edges $\omega \in \{M_{S(n)}(0) \ge 3\}$, we pick $x = x(\omega)$, $y = y(\omega)$ and $z = z(\omega)$ and the three paths as just described. Let $J_{x,y,z}$ be the event that all edges in these three paths are open and that all other edges in $S(n)$ are closed. Then

$$\mathbb{P}_p(J_{x,y,z} \mid M_B(0) \ge 3) \ge (\min\{p, 1-p\})^{R(n)}$$

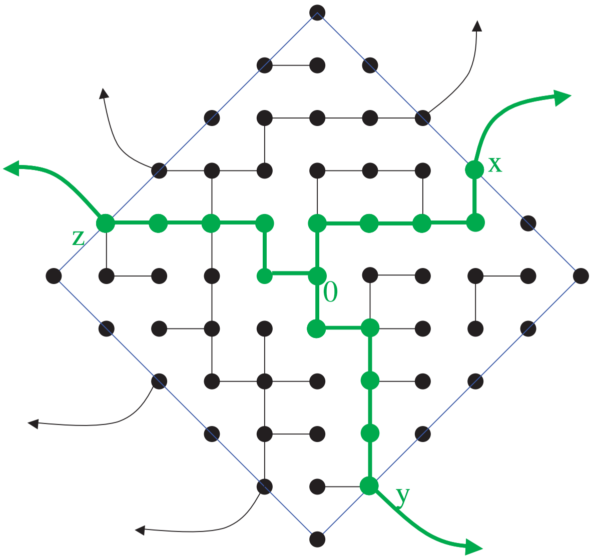


Figure 8.10: A sufficient condition for 0 to be a trifurcation if the three paths from x, y and z are open, and all all other edges in $S(n)$ are closed, and x, y and z belong to three distinct infinite open clusters. The arrows outside $\partial S(n)$ represent connectivity to distinct infinite clusters.

where $R(n)$ is the total number of edges in $S(n)$. Now, if $M_{S(n)}(0) \geq 3$ and if $J_{x,y,z}$ occurs, then x is a trifurcation. Therefore

$$\begin{aligned} \mathbb{P}_p(0 \text{ is a trifurcation}) &\geq \mathbb{P}_p(J_{x,y,z} \mid M_B(0) \geq 3)\mathbb{P}_p(M_{S(n)}(0) \geq 3) \\ &\geq (\min\{p, 1-p\})^{R(n)} /2 > 0. \end{aligned}$$

Because of (8.11), we have therefore that $\mathbb{P}_p(x \text{ is a trifurcation}) > 0$ for all vertices $x \in \mathbb{Z}^d$. Let $T(m)$ denote the number of trifurcations in $S(m)$. As

$$\mathbb{E}_p[T(m)] = |S(m)|\mathbb{P}_p(0 \text{ is a trifurcation}),$$

it implies that $T(m)$ grows in the manner of $|S(m)|$ as $m \to \infty$. The contradiction is obtained by the following rough geometric argument (a more rigorous proof uses partitions, see [7]). Pick a trifurcation in $S(m)$, say t_1, and take a vertex $x_1 \in \partial S(m)$ that is connected to t_1 by an open path in $S(m)$. Pick a second trifurcation $t_2 \in S(m)$. By definition of a trifurcation, there must be a vertex $x_2 \in \partial S(m)$, distinct from x_1, such that $t_2 \leftrightarrow x_2$ in $S(m)$ (See Figure 8.11). Repeat this operation, at each stage picking a new trifurcation t_i and a new vertex $x_i \in \partial S(m)$, with $t_i \leftrightarrow x_i$ in $S(m)$. There are $T(m)$ trifurcations in $S(m)$, so we end up with $T(m)$ distinct vertices $x_i \in \partial S(m)$, which implies that $|\partial S(m)| \geq T(m)$. But $T(m)$ grows in the manner of $|S(m)|$ for large m, which would mean that $|\partial S(m)|$ would grow in the manner of $|S(m)|$ for large m as well. We have reached a contradiction, because $|S(m)|$ grows in the manner of m^d while $|\partial S(m)|$ grows in the manner of m^{d-1}.

■

8.5.2 Finite cluster size distribution

We are now interested in the size of the finite clusters. We only consider the 2-dim case, where the proof is easier because of the use of duality. The theorem is however valid for $d \geq 2$.

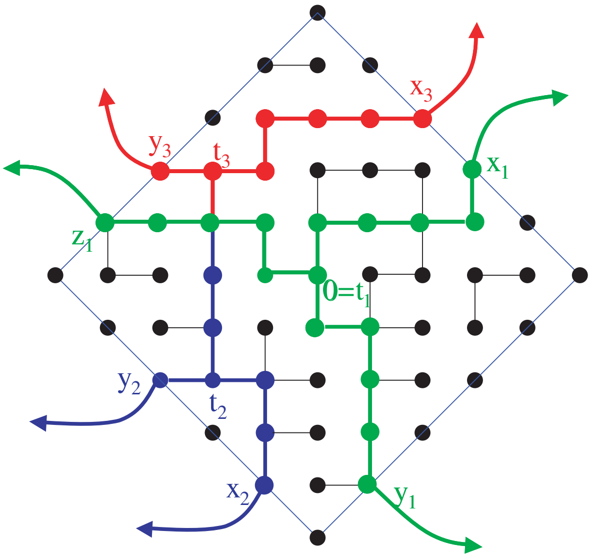


Figure 8.11: Finding trifurcations in $S(n)$.

Theorem 8.8 (Sub-exponential decay of the finite cluster size distribution)**.** *If $p_c < p < 1$, there exists $\eta(p) > 0$ such that*

$$\mathbb{P}_p(|C| = n) \leq \exp(-n^{(d-1)/d}\eta(p)) \tag{8.12}$$

for $n \in \mathbb{N}^*$.

We can also find a lower bound of the same form: there exists some $\gamma(p) < \infty$ such that

$$\mathbb{P}_p(|C| = n) \geq \exp(-n^{(d-1)/d}\gamma(p)).$$

Proof:

We only prove a slightly weaker bound

$$\mathbb{P}_p(|C| = n) \leq n \exp(-\sqrt{n}\eta(p)),$$

and only for $d = 2$ and for $2/3 < p < 1$. Once we will have computed the exact value of p_c in the next chapter, the proof is directly extended for $p_c < p < 1$.

Suppose that that the origin belongs to a finite cluster of size n. Then there exists a closed circuit in the dual lattice $\mathbb{L}_d^2$, having the origin in its interior. Clearly, this circuit has less than n vertices. Moreover, it can be shown using topological arguments (see Kesten 1982) that there is some value $\delta > 0$ such that this closed circuit contains at least $\delta\sqrt{n}$ vertices. For the same reason as in part (ii) of the proof of Theorem 8.1, it must pass through a vertex of the form $(i + 1/2, 1/2)$ for some $0 \leq i \leq n-1$, and therefore one of these n vertices must lie in a closed cluster of $\mathbb{L}_d^2$ of size at least $\delta\sqrt{n}$. Let us called 0_d this vertex, and C_d the closed cluster to which it belongs.

Now, each edge of $\mathbb{L}_d^2$ is closed with probability $(1-p)$, and $1-p < 1/3 \leq p_c$ because of Theorem 8.1. In other words, the process of closed edges of $\mathbb{L}_d^2$ is subcritical. Theorem 8.6 then yields that there exists $\lambda(p) > 0$ such that

$$\mathbb{P}_p(|C_d| \geq \delta\sqrt{n}) \leq \exp(-\lambda(p)\delta\sqrt{n}).$$

Since

$$\mathbb{P}_p\left((i + 1/2, 1/2) \text{ lies in a closed cluster of } \mathbb{L}_d^2 \text{ of size at least } \delta\sqrt{n}\right) = \mathbb{P}_p(|C_d| \geq \delta\sqrt{n}),$$

we have thus that

$$\begin{aligned}\mathbb{P}_p(|C|=n) &\leq \sum_{i=0}^{n-1}\mathbb{P}_p\left((i+1/2,1/2) \text{ lies in a closed cluster of } \mathbb{L}_d^2 \text{ of size at least } \delta\sqrt{n}\right)\\ &= n\mathbb{P}_p(|C_d|\geq \delta\sqrt{n})\\ &\leq n\exp(-\lambda(p)\delta\sqrt{n}).\end{aligned}$$

Setting $\eta(p)=\lambda(p)\delta$ finishes the proof.

■

8.6 Near the Critical Threshold

After having studied the key properties of the metrics associated to the cluster size distribution in the sub-critical phase, we now move to the critical point p_c.

8.6.1 Critical threshold for bond percolation on the 2-dim. lattice

The previous sections have equipped us with the necessary tools to eventually compute the value of p_c, which we will prove to be equal to 1/2.

We begin by proving that in 2 dimensions, the percolation probability is zero when $p=1/2$. An immediate consequence is that the critical percolation threshold $p_c \geq 1/2$.

The absence of infinite open cluster at the percolation threshold is also conjectured to hold in higher dimensions.

Lemma 8.3 (Absence of infinite open cluster for $p=1/2$)**.** *If $d=2$, then $\theta(1/2)=0$.*

Proof:

We proceed by contradiction, and follow Zhang (1988) as exposed in [7]. Suppose that $\theta(1/2)>0$. Consider the square $B(n)=[-n,n]\times[-n,n]$, and let $A^l(n)$ (respectively, $A^r(n)$, $A^t(n)$, $A^b(n)$) be the event that some vertex on the left (respectively, right, top, bottom) side of $B(n)$ belongs to an infinite open path of $\mathbb{L}^2$ that uses no other vertex of $B(n)$. Clearly, these are four increasing events that have equal probability (by symmetry) and whose union is the event that some vertex on $B(n)$ belongs to an infinite cluster. Since we assume that $\theta(1/2)>0$, the Kolmogorov zero-one law implies that there is almost surely an infinite cluster, and therefore as $n\to\infty$,

$$\mathbb{P}_{1/2}\left(A^l(n)\cup A^r(n)\cup A^t(n)\cup A^b(n)\right)\to 1. \tag{8.13}$$

Now, using the "square root trick", which states that if B_i, $1\leq i\leq n$, are increasing events having the same probability,

$$\mathbb{P}_p(B_i)\geq 1-\left(1-\mathbb{P}_p\left(\bigcup_{i=1}^{n}B_i\right)\right)^{1/n},$$

we get

$$\mathbb{P}_{1/2}(A^l(n))\geq 1-\left(1-\mathbb{P}_{1/2}\left(A^l(n)\cup A^r(n)\cup A^t(n)\cup A^b(n)\right)\right)^{1/4}.$$

It follows from (8.13) that, with $u=l,r,t,b$,

$$\mathbb{P}_{1/2}(A^u(n))\to 1$$

as $n \to \infty$. Therefore there is n_0 large enough such that for $u = l, r, t, b$

$$\mathbb{P}_{1/2}(A^u(n_0)) > 7/8. \tag{8.14}$$

Let us next consider the dual box $B_d(n)$ defined as

$$B_d(n) = \{(i + 1/2, j + 1/2) \mid (i, j) \in B(n)\},$$

and let $A^l_d(n)$ (respectively, $A^r_d(n)$, $A^t_d(n)$, $A^b_d(n)$) be the event that some vertex on the left (respectively, right, top, bottom) side of $B_d(n)$ belongs to an infinite closed path of $\mathbb{L}^2_d$ that uses no other vertex of $B_d(n)$. Each edge of $\mathbb{L}^2_d$ is closed with a probability 1/2, which is the same as the open edge probability in $\mathbb{L}^2$. Therefore $\mathbb{P}_{1/2}(A^u(n)) = \mathbb{P}_{1/2}(A^u_d(n))$ for $u = l, r, t, b$ and all $n \in \mathbb{N}^*$. In particular, for $n = n_0$,

$$\mathbb{P}_{1/2}(A^u_d(n_0)) > 7/8 \tag{8.15}$$

for $u = l, r, t, b$ because of (8.14).

We now consider the event $A = A^l(n_0) \cap A^r(n_0) \cap A^t_d(n_0) \cap A^b_d(n_0)$, that there exist infinite open paths of $\mathbb{L}^2$ connecting to some vertex on the left and right sides of $B(n)$, without using any other vertex of $B(n)$, and that there exists infinite closed paths connecting to some vertex on the top and bottom sides of $B_d(n)$, without using any other vertex of $B_d(n)$, as shown in Figure 8.12. Now, using the union bound,

$$\begin{aligned}
\mathbb{P}_{1/2}(A) &= 1 - \mathbb{P}_{1/2}\left(\overline{A}^l(n_0) \cup \overline{A}^r(n_0) \cup \overline{A}^t_d(n_0) \cup \overline{A}^b_d(n_0)\right) \\
&\geq 1 - \left(\mathbb{P}_{1/2}(\overline{A}^l(n_0)) + \mathbb{P}_{1/2}(\overline{A}^r(n_0)) + \mathbb{P}_{1/2}\overline{A}^t_d(n_0)) + \mathbb{P}_{1/2}(\overline{A}^b_d(n_0))\right) \\
&> 1/2
\end{aligned}$$

because of (8.14) and (8.15). If A occurs, then there must be two infinite open clusters in $\mathbb{L}^2 \setminus B(n_0)$, one containing the infinite open path connected to the left side of $B(n)$ and the other one containing the infinite open path connected to the right side of $B(n)$. Moreover, these two infinite open clusters must be disjoint, because they are separated by two infinite closed paths in $\mathbb{L}^2_d \setminus B_d(n_0)$ connecting to some vertex on the top and bottom sides of $B_d(n)$. If there was an open path connecting the two infinite clusters of $\mathbb{L}^2 \setminus B(n_0)$ path, one of its (open) edges would cross a closed edge in $\mathbb{L}^2_d \setminus B_d(n_0)$, which is impossible, as shown in Figure 8.12.

The same reasoning implies that there must be two disjoint infinite closed clusters in $\mathbb{L}^2_d \setminus B_d(n_0)$, one containing the infinite closed path connected to the top side of $B_d(n)$ and the other one containing the infinite closed path connected to the bottom side of $B_d(n)$, and separated by the two infinite open paths of $\mathbb{L}^2 \setminus B(n_0)$. Now, as $\theta(1/2) > 0$, Theorem 8.7 yields that the infinite lattice $\mathbb{L}^2$ contains (almost surely) one and only one infinite open cluster. Therefore, there must be a left-right open crossing within $B(n)$, which forms a barrier to any top-bottom closed crossing of $B_d(n)$. As a result, there must be (almost surely) at least two disjoint infinite closed clusters in $\mathbb{L}^2_d$. But since $p = 1 - p = 1/2$, the probability that there are two infinite closed clusters in $\mathbb{L}^2_d$ is the same as the probability that there are two infinite open clusters in $\mathbb{L}^2$, which is zero. We have thus reached a contradiction, which means that $\mathbb{P}_{1/2}(A)$ cannot be nonzero. The initial assumption $\theta(1/2) > 0$ cannot be valid, which establishes the result.

■

The previous theorem implies that $p_c \geq 1/2$. The following lemma is the main step in showing the converse, namely $p_c \leq 1/2$.

Lemma 8.4 (Crossing of a square for $p = 1/2$). *Let $LR(n)$ be the even that there is a left-right open crossing of the rectangle $R(n) = [0, n+1] \times [0, n]$ (that is, an open path connecting some vertex on the left side of $R(n)$ to some vertex on the right side of $R(n)$. Then $\mathbb{P}_{1/2}(LR(n)) = 1/2$ for all $n \in \mathbb{N}^\star$.*

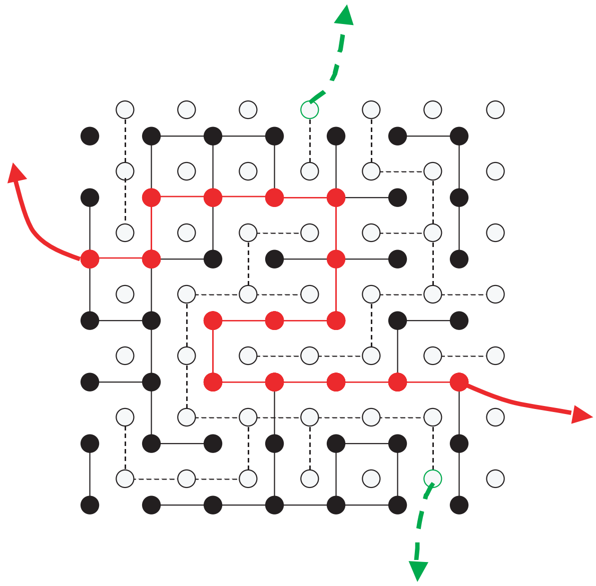


Figure 8.12: Infinite open paths of $\mathbb{L}^2 \setminus B(n_0)$ connecting to some vertex on the left and right sides of $B(n_0)$ and infinite closed paths of $\mathbb{L}_d^2 \setminus B_d(n_0)$ connecting to some vertex on the top and bottom sides of $B_d(n)$.

Proof:

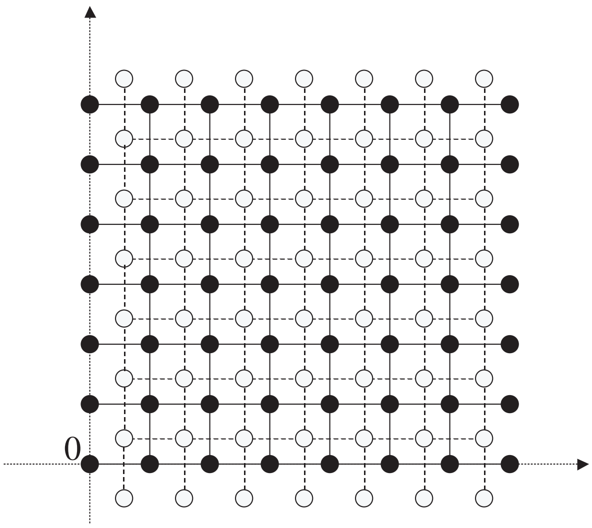
The rectangle $R(n)$ is the subgraph of $\mathbb{L}^2$ having vertex set $[0, n+1] \times [0, n]$ and edge set comprising all edges of $\mathbb{L}^2$ joining pairs of vertices in $S(n)$, except those joining pairs (i, j), (k, l) with either $i = k = 0$ or $i = k = n+1$. Let $R_d(n)$ be the subgraph of $\mathbb{L}_d^2$ having vertex set $\{(i + 1/2, j + 1/2) \mid 0 \le i \le n, 1 \le j \le n\}$ and edge set all edges of $\mathbb{L}_d^2$ joining pairs of vertices in $R_d(n)$, except those joining pairs (i, j), (k, l) with either $i = k = -1/2$ or $i = k = n+1/2$. The two subgraphs can be obtained from each other by a 90 degrees rotation, which relocates the vertex labeled $(0, 0)$ at the point $(n + 1/2, -1/2)$, see Figure 8.13 (left).

Let us consider the two following events: $LR(n)$ is the event that there exists an open path of $R(n)$ joining a vertex on the left side of $R(n)$ to a vertex on its right side, and $TB_d(n)$ is the event that there exists a closed path of $R_d(n)$ joining a vertex on the top side of $R_d(n)$ to a vertex on its bottom side.

If $LR(n) \cap TB_d(n) \neq \emptyset$, there is a left-right open path in $R(n)$ crossing a top-bottom closed path in $S_d(n)$. But then, at the crossing of these two paths, there would be an open edge of $\mathbb{L}^2$ crossed by a closed edge of $\mathbb{L}_d^2$, which is impossible, see Figure 8.13 (right). Hence $LR(n) \cap TBd(n) = \emptyset$. On the other hand, either $LR(n)$ or $TBd(n)$ must occur. Let D be the set of vertices that are reachable from the left side of $R(n)$ by an open path. Suppose that $LR(n)$ does not occur. Then there exists a top-bottom closed path of $\mathbb{L}_d^2$ crossing only edges of $R(n)$ contained in the edge boundary of D, and so $TBd(n)$ occurs. Consequently $LR(n)$ and $TBd(n)$ form a partition of the sample space Ω, and

$$\mathbb{P}_p(LR(n)) + \mathbb{P}_p(TBd(n)) = 1. \tag{8.16}$$

Now, since $R(n)$ and $R_d(n)$ are isomorphic (they can be obtained from each other by a 90 degrees rotation, which relocates the vertex labeled $(0, 0)$ at the point $(n+1/2, -1/2)$), flipping the polarity of each edge of $\mathbb{L}_d^2$ yields that $\mathbb{P}_p(TBd(n)) = \mathbb{P}_{1-p}(LR(n))$. Plugging this equality in (8.16), the



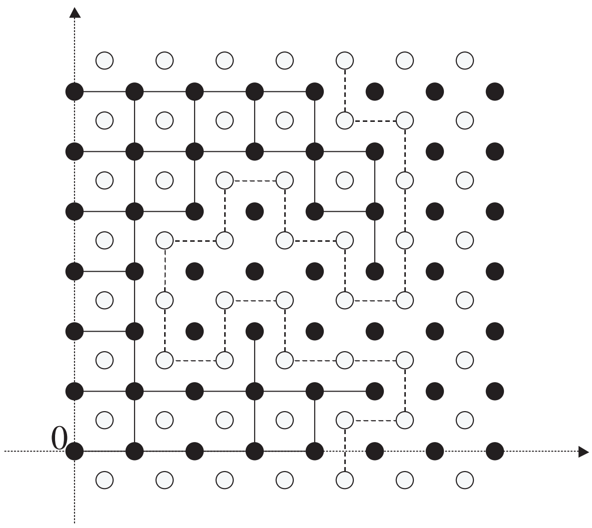


Figure 8.13: The box $R(n)$ and its dual $R_d(n)$ for $n = 6$ (left) and an illustration of the fact that there is no left-right open crossing of $R(n)$ is and only if there is a top-bottom closed crossing of $R_d(n)$ (right).

latter becomes

$$\mathbb{P}_p(LR(n)) + \mathbb{P}_{1-p}(LR(n)) = 1.$$

Taking $p = 1/2$ in this equation proves the lemma.

■

We now deduce directly one of the most famous theorems of percolation theory.

Theorem 8.9 ($p_c = 1/2$)**.** *The percolation threshold in $\mathbb{L}^2$ is $p_c = 1/2$.*

Proof:

We know from Lemma 8.3 that $p_c \geq 1/2$. Suppose that $p_c > 1/2$. Then the value $p = 1/2$ belongs to the subcritical phase, and we know from Theorem 8.4 that there exists $\psi(1/2) > 0$ such that for all $n \in \mathbb{N}^*$

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial^r R(n)) \leq \mathbb{P}_{1/2}(0 \leftrightarrow \partial S(n)) < \exp(-n\psi(1/2)),$$

where $\{0 \leftrightarrow \partial^r R(n)\}$ is the event that the origin is connected by an open path to a vertex lying on the right side of $R(n)$, defined as $\partial^r R(n) = \{(n+1, k) \in \mathbb{Z}^2 \mid 0 \leq k \leq n\}$, and where $\{0 \leftrightarrow \partial S(n)\}$ is the event that the origin is connected by an open path to a vertex lying on the perimeter of the ball of radius n centered in 0. Consequently, since $LR(n)$ is the event that there exists an open path of $R(n)$ joining a vertex on the left side of $R(n)$ to a vertex on its right side,

$$\begin{aligned}
\mathbb{P}_{1/2}(LR(n)) &\leq \sum_{k=0}^{n} \mathbb{P}_{1/2}((0,k) \leftrightarrow \partial^r R(n)) \\
&\leq (n+1)\mathbb{P}_{1/2}(0 \leftrightarrow \partial^r R(n)) \\
&< (n+1)\exp(-n\psi(1/2)),
\end{aligned}$$

which yields that $P_{1/2}(LR(n)) \to 0$ as $n \to \infty$, and therefore contradicts Lemma 8.4. Consequently $p_c \leq 1/2$, which completes the proof.

■

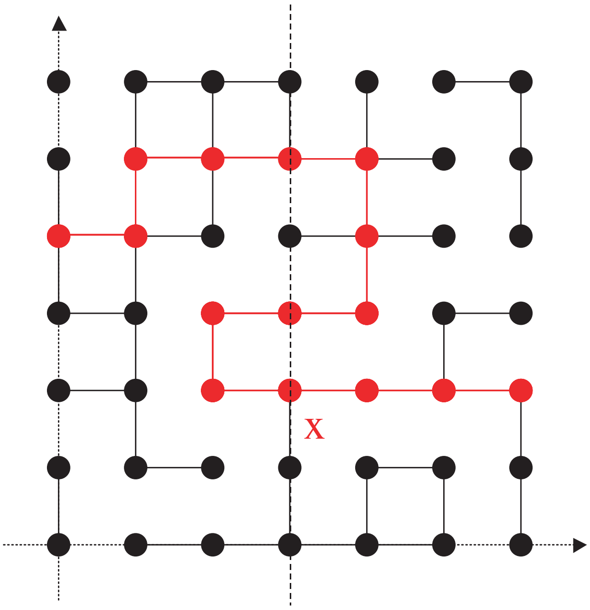


Figure 8.14: A left right open path crossing the box $R(2n-1)$ must hit the center vertical line at some vertex x, which is therefore joined by two disjoint paths to respectively the left and right sides of $R(2n-1)$.

8.6.2 Near the Critical Threshold: Power Laws

We know from Theorems 8.4 and 8.6 that the distributions of the radius and size of the cluster at the origin C decrease exponentially fast when $p < 1/2$. What happens when $p = 1/2$? Lemma 8.3 indicates that the cluster C is almost surely finite at the critical threshold, like in the sub-critical phase. The following theorem shows however that the distributions of the radius and size change radically of nature, and are follow no longer an exponential law, but a power law. A consequence is that the mean cluster size $\chi(1/2) = \infty$, contrary to the subcritical case.

Theorem 8.10 (Power law inequalities at the critical threshold)**.** *In $\mathbb{L}^2$, for all $n \in \mathbb{N}^*$,*

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial B(n)) \geq \frac{1}{2\sqrt{n}} \tag{8.17}$$

$$\mathbb{P}_{1/2}(|C| \geq n) \geq \frac{1}{2\sqrt{n}}. \tag{8.18}$$

Proof:

Since any open path connecting the origin to the perimeter of $B(n)$ contains at least n vertices, $\mathbb{P}_{1/2}(|C| \geq n) \geq \mathbb{P}_{1/2}(0 \leftrightarrow \partial B(n))$, and so we only need to prove (8.17).

As before, let $LR(2n-1)$ be the event that is an open path in the rectangle $R(2n-1) = [0, 2n] \times [0, 2n-1]$ connecting some vertex on its left side to some vertex on its right side. This path must cross the center line $\{(n,k) \in \mathbb{Z}^2 \mid 0 \leq k \leq 2n-1\}$ in at least one vertex, which is therefore connected by two disjoint paths to respectively the left and right sides of $R(2n-1)$, as shown in Figure 8.14.

Denoting by $A_n(k)$ the event that the vertex (n,k) is joined by an open path to the surface

$\partial B(n,(n,k))$ of the box $B(n,(n,k))$ having side-length $2n$ and centered at (n,k), we have therefore

$$\mathbb{P}_{1/2}(LR(2n-1)) \leq \sum_{k=0}^{2n-1} \mathbb{P}_{1/2}(A_n(k) \circ A_n(k))$$

and applying the BK inequality, we get

$$\begin{aligned}\mathbb{P}_{1/2}\left(LR(2n-1)\right) &\leq \sum_{k=0}^{2n-1} \mathbb{P}^2_{1/2}(A_n(k)) \\ &= 2n\mathbb{P}^2_{1/2}(A_n(0)) \\ &= 2n\mathbb{P}^2_{1/2}(0 \leftrightarrow \partial B(n)).\end{aligned}$$

Now, Lemma 8.4 states that $\mathbb{P}_{1/2}\left(LR(2n-1)\right) = 1/2$ for all $n \in \mathbb{N}^*$, from which we deduce (8.17).

■

We obtain directly that the tail of distribution of the radius $\mathrm{rad}(C) = \max_{x\in C}\{|x|\}$ of the cluster size C from (8.17) by noting that

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n/2)) \leq \mathbb{P}_p(0 \leftrightarrow \partial S(n)) = \mathbb{P}_p(\mathrm{rad}(C) \geq n).$$

8.7 Appendix: Kolmogorov's zero-one law and tail events

Let $\{X_n, n \in \mathbb{N}^*\}$ be a sequence of independent random variables. A *tail event* is an event whose occurrence or failure is determined by the values of these random variables, but which does not depend probabilistically of any finite subsequence of these random variables.

For example, the event $\{\sum_{n\in\mathbb{N}^*} X_n$ converges $\}$ is a tail event, because if we remove any finite subcollection of X_n, it does not change the convergence property of the series. Likewise, the event $\{\limsup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} X_m \leq 2\}$ is a tail event. On the contrary, for if $\sum_{n\in\mathbb{N}^*} X_n$ converges, the event $\{\sum_{n\in\mathbb{N}^*} X_n \leq 2\}$ does change if we remove some finite subcollection of X_n, and thus is not a tail event.

In the case of Corollary 8.1, let X_n denotes the state of an edge and A be the existence of an infinite open cluster. Then A does not depend on any finite subcollection of variables X_n, and is therefore a tail event.

Tail events enjoy the following property.

Theorem 8.11 (Kolmogorov's zero-one law)**.** *If $\{X_n, n \in \mathbb{N}^*\}$ is a sequence of independent variables, then any tail event A satisfies $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.*

The following corollary of the zero-one law will be useful later on (see [8]). Let Y be a random variable which is a function of the variables X_n. Then Y is a *tail function* if, roughly speaking, it does not depend crucially on any finite subcollection of X_n. More precisely, Y is a tail function if and only if the event $\{\omega \in \Omega \mid Y(\omega) \leq y\}$ is a tail event for all $y \in \mathbb{R}$.

For example, the random variable

$$Y = \lim\sup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} X_n$$

is a tail function of the independent variables X_n.

Tail functions are almost surely constant. Indeed, since $\{\omega \in \Omega \mid Y(\omega) \leq y\}$ is a tail event for all $y \in \mathbb{R}$, $\mathbb{P}(Y \leq y)$ can only take the values 0 and 1. Let $k = \inf\{y \mid \mathbb{P}(Y \leq y) = 1\}$. Then for any $y \in \mathbb{R}$, $\mathbb{P}(Y \leq y) = 0$ when $y < k$ and $\mathbb{P}(Y \leq y) = 1$ when $y \geq k$.

Theorem 8.12 (Constant tail functions)**.** *If Y is a tail function of the independent variables $X_n, n \in \mathbb{N}^*$, then there exists some $k \in \mathbb{Z} \cup \{-\infty, \infty\}$ such that $\mathbb{P}(Y = k) = 1$.*

9

Discrete Percolation Models

9.1 Introduction

We have seen two discrete percolation models on the tree and the lattice. Tree percolation is the simplest percolation model, because circuits are absent and there is now a unique path between any two vertices of the tree. Bond percolation on $\mathbb{L}^d$ is a canonical representative of the class of percolation models, and most result for this model are still valid for other, more general models. We will complete our study of discrete percolation models by sitepercolation, which is at the same time more difficult to handle than bond percolation, because it amounts to introduce dependencies between the edges, but also more general, because to every bond model corresponds a site model, but not vice-versa. Nevertheless, most findings of bond percolation carry over at least qualitatively to site percolation.

Before investigating the site percolation model, we will first examine some striking similarities between bond and tree percolation, which lead physicists to consider universal scaling laws valid on different graph topologies.

9.1.1 Scaling theory

Scaling theory has been used by mathematical physicists to study the behavior of quantities such as $\theta(p)$, $\xi(p)$, $\mathbb{P}_p(|C| = n)$ near the critical point p_c.

For bond percolation in $\mathbb{L}^2$, Theorem 8.10 suggests that they follow a power law distribution in that transition region, and indeed this is taken as starting assumption (ansatz in Physics) for such quantities. In other words, scaling theory assumes that

$$\begin{aligned}
\theta(p) &\approx (p - p_c)^{\beta} && \text{as } p \downarrow p_c && (9.1)\\
\chi(p) &\approx (p - p_c)^{-\gamma} && \text{as } p \uparrow p_c && (9.2)\\
\xi(p) &\approx (p - p_c)^{-\nu} && \text{as } p \uparrow p_c && (9.3)\\
\mathbb{P}_{p_c}(|C| = n) &\approx n^{-1-1/\delta} && \text{as } n \to \infty && (9.4)\\
\mathbb{P}_{p_c}(\mathrm{rad}(C) \geq n) &\approx n^{-1/\rho} && \text{as } n \to \infty && (9.5)
\end{aligned}$$

where the "critical exponents" $\beta > 0$, $\gamma > 0$, $\nu > 0$, $\delta > 1$ and $\rho > 0$ depend on the dimension d. The notation $f(p) \approx g(p)$ as $p \to p_c$ means that $\lim_{p \to p_c} \ln f(p)/\ln g(p) = 1$.

We from Chapter 2 that (9.1), (9.2) and (9.4) apply to tree percolation on the binary tree $\mathbb{T}^2$ as well, with $\beta = 1$, $\gamma = 1$ and $\delta = 2$. We could extend the computations to include (9.3) and (9.5) as well.

Scaling theory predicts that these critical exponents are not independent from each other, but obey sets of relations called "scaling relations". We are going to derive the one linking β, γ and δ.

More precisely, the ansatz for the distribution of the cluster at the origin for values for $p \leq p_c$ is

$$\mathbb{P}_p(|C| = n) = n^{-(1+\delta^{-1})} f_-\left(n/\xi^{\tau}(p)\right) \tag{9.6}$$

where $\tau > 0$ is a constant, and $f_-(\cdot)$ is a smooth (differentiable) positive function on $\mathbb{R}^+$. Theorem 8.6 would suggest a function $f_-(x) \approx C\exp(-Ax)$ for some $A, C > 0$, but this is not very important here. We just assume that $f_-(x) \to 0$ faster than any power of $1/x$ as $x \to \infty$.

When $p \geq p_c$, we will take a similar ansatz for the distribution of the finite cluster at the origin

$$\mathbb{P}_p(|C| = n) = n^{-(1+\delta^{-1})} f_+\left(n/\xi^{\tau}(p)\right) \tag{9.7}$$

where $f_+(\cdot)$ is a smooth (differentiable) positive function on $\mathbb{R}^+$. Here again, Theorem 8.8 would suggest to take $f_+(x) \approx C'\exp(-A'x^{(d-1)/d})$ for some $A', C' > 0$, but again we do not want to make this assumption here. We just assume that $f_+(x) \to 0$ faster than any power of $1/x$ as $x \to \infty$.

Now, we make the following approximate computations, first when $p < p_c$:

$$\begin{aligned}
\chi(p) &= \sum_{n \in \mathbb{N}^*} n\mathbb{P}_p(|C| = n) \simeq \sum_{n \in \mathbb{N}^*} n^{-\delta^{-1}} f_-\left(n/\xi^{\tau}(p)\right)\\
&\simeq \int_0^{\infty} n^{-\delta^{-1}} f_-\left(n/\xi^{\tau}(p)\right) dn\\
&= \xi^{\tau(1-\delta^{-1})}(p) \int_0^{\infty} u^{-\delta^{-1}} f_-(u) du.
\end{aligned}$$

Making the assumptions (9.2) and (9.3), the latter equation becomes

$$(p - p_c)^{-\gamma} \approx (p - p_c)^{-\nu\tau(1-\delta^{-1})} \int_0^{\infty} u^{-\delta^{-1}} f_-(u) du,$$

and since the integral converges because we assumed $\delta > 1$, we find that

$$\tau\nu = \frac{\gamma}{1 - \delta^{-1}} \tag{9.8}$$

We continue now with $p > p_c$, assuming that $\theta(p_c) = 0$ (we know it for sure for $d = 2$), so that

$$\begin{aligned}
\theta(p) &= 1 - \sum_{n \in \mathbb{N}^*} \mathbb{P}_p(|C| = n) \\
&= \sum_{n \in \mathbb{N}^*} \left[\mathbb{P}_{p_c}(|C| = n) - \mathbb{P}_p(|C| = n)\right] \\
&\simeq \sum_{n \in \mathbb{N}^*} n^{-1-\delta^{-1}} \left[f_+(0) - f_+\left(n/\xi^\tau(p)\right)\right] \\
&\simeq \int_0^\infty n^{-1-\delta^{-1}} \left[f_+(0) - f_+\left(n/\xi^\tau(p)\right)\right] dn \\
&= \xi^{-\tau\delta^{-1}}(p) \int_0^\infty u^{-1-\delta^{-1}} \left[f_+(0) - f_+(u)\right] du
\end{aligned}$$

Plugging (9.1) and (9.3) in the latter equation, it becomes

$$(p - p_c)^\beta \approx (p - p_c)^{\nu\tau\delta^{-1}} \int_0^\infty u^{-1-\delta^{-1}} \left[f_+(0) - f_+(u)\right] du.$$

The integrand behaves like $u^{-1-\delta^{-1}} df_+(0)/du$ near $u = 0$, hence the integral converges. As a result, we find that

$$\tau\nu = \beta\delta.$$

Combining this relation with (9.8) gives the scaling relation

$$\gamma + \beta = \beta\delta. \tag{9.9}$$

This latter equation is one among many scaling relations. It shows that at least one among the three critical exponents, at most two are independent from each other. Interestingly, if relation (9.9) does not depend on the dimension $d \geq 2$ of the lattice $\mathbb{L}^d$. We also observe that the three values $\beta = 1$, $\gamma = 1$ and $\delta = 2$ that we computed in Chapter 2 for the binary tree $\mathbb{T}^2$ satisfy the scaling relation (9.9) as well!

Another set of relations depend on the dimension d, and are believed to be valid only for dimensions $2 \leq d \leq d_c$ where the d_c is called the "critical dimension". These relations are called hyper-scaling relations, and read

$$d\nu = \gamma + 2\beta \tag{9.10}$$

$$d\rho = \delta + 1. \tag{9.11}$$

The scaling relations are widely accepted, but the hyper-scaling relations are more questionable.

The values of the scaling exponents obtained numerically for $\mathbb{L}^2$ are $\beta = 5/36$, $\gamma = 43/18$, $\delta = 91/5$, $\nu = 4/3$, $\rho = 48/5$.

The values of the scaling exponents for $\mathbb{T}^2$ are $\beta = 1$, $\gamma = 1$, $\delta = 2$ (as we saw in Chapter 2), $\rho = 1/2$ and $\nu = 1/2$. Plugging these values in the hyperscaling relations (9.10) and (9.11), we find that $d = 6$. This suggests that the critical dimension $d_c = 6$. Indeed, we can embed $\mathbb{T}^2$ in $\mathbb{L}^\infty$, with each edge connecting a nth layer vertex of $\mathbb{T}^2$ to a $(n+1)$th layer vertex being parallel to the nth coordinate axis of $\mathbb{L}^\infty$. This would make percolation in $\mathbb{T}^2$ and $\mathbb{L}^\infty$ similar, roughly speaking. The computations for the tree suggests that the two processes are similar already for $\mathbb{L}^d$ with $d \geq 6$.

9.2 Site percolation

An important model is to close vertices rather than edges in a lattice $\mathbb{L}^d$. The corresponding model is called site percolation, all definitions of percolation probability, critical probability, etc remain

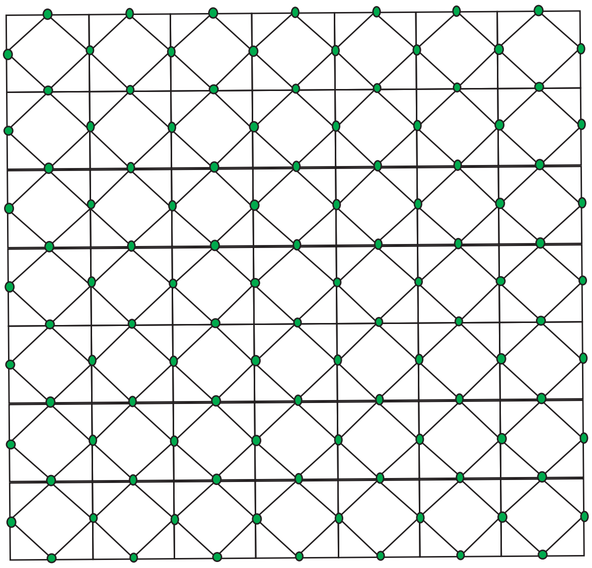


Figure 9.1: The covering graph $\mathbb{L}_c^2$ of the square lattice $\mathbb{L}^2$.

the same as in the bond model, the only difference is that vertices (and not edges) are open with probability p, and closed with probability $1 - p$.

One can show that a phase transition occurs between a sub-critical and super-critical phases, and essentially most properties of bond percolation extend to site percolation. However, contrary to bond percolation, the percolation threshold p_c is not known mathematically. It is found numerically to be close to 0.59.

Site percolation is more general than bond percolation, in the sense that every bond model can be recast as a site model, but not the reverse. To recast a bond model as a site model, we make use of the notion of covering graph $\mathcal{G}_c$ of a graph $\mathcal{G}$, which is obtained as follows. Place a vertex of $\mathcal{G}_c$ on the middle of each edge of $\mathcal{G}$. Two vertices of $\mathcal{G}_c$ are declared to be adjacent if and only if the two corresponding edges of $\mathcal{G}$ share a common endvertex of $\mathcal{G}$. Define now a bond percolation process on $\mathcal{G}$, and declare a vertex of $\mathcal{G}_c$ to be open (resp., closed) if and only if the corresponding edge of $\mathcal{G}$ is open (resp., closed). This results in a site percolation process on $\mathcal{G}_c$. Any path of open edges of $\mathcal{G}$ corresponds to a path of open vertices of $\mathcal{G}_c$, and vice-versa. As a result,

$$p_c^{\text{bond}}(\mathcal{G}) = p_c^{\text{site}}(\mathcal{G}_c) \tag{9.12}$$

For example, if $\mathcal{G} = \mathbb{L}^2$, then we find that $\mathcal{G}_c = \mathbb{L}_c^2$ is the lattice shown in Figure 9.1, where each site has exactly six adjacent vertices. Because of (9.12), the site percolation threshold on this graph is 1/2. We can show that the triangular lattice, where each vertex is also adjacent to six other vertices, has a also a site percolation threshold equal to 1/2.