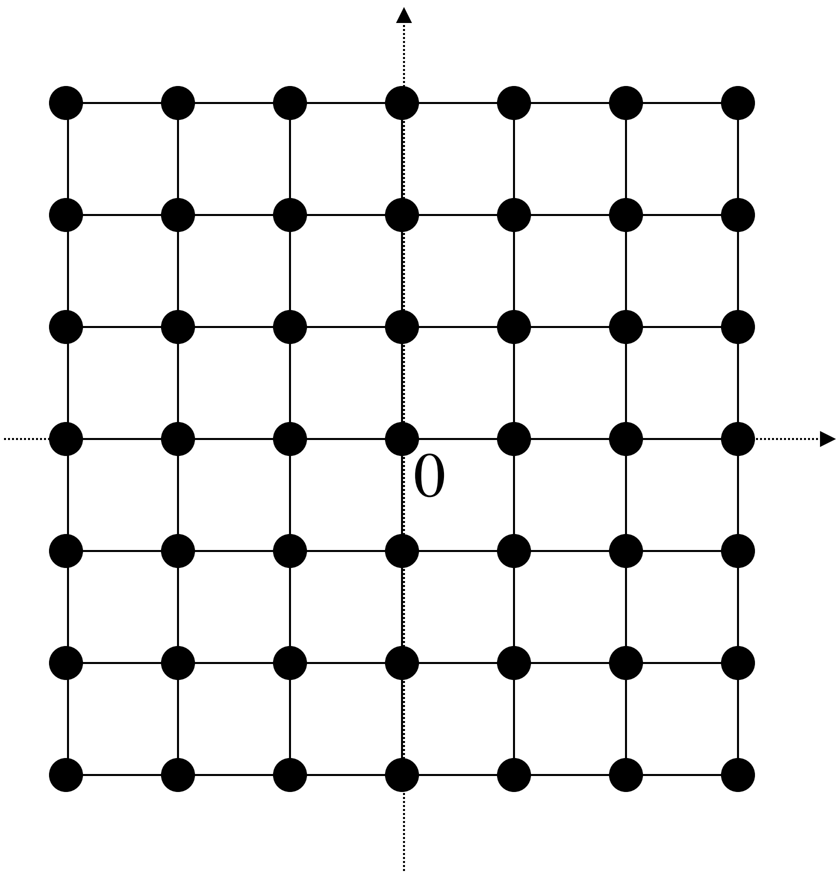
The main questions

- Prove that there is a phase transition
- Sub-critical phase: no infinite cluster ($\theta(p) = P(|C| = \infty) = 0$) , but
  - Is the mean cluster size $\chi(p) = E[|C|]$ finite ?
  - What is the tail of the distribution of C: $P(|C| = n)$ for large n ?
- Super-critical phase: infinite cluster ($\theta(p) = P(|C| = \infty) > 0$)
  - Is the infinite cluster unique ?
  - If so, what is the tail of the distribution of the second largest cluster C^f: $P(|C^f| = n)$ for large n ?
- What is the critical threshold p_c ?
- What happens when $p = p_c$, or at least when $|p - p_c|$ is small:
  - What is $\theta(p) = P(|C| = \infty)$?
  - Is the mean cluster size $\chi(p) = E[|C|]$ finite ?
  - What is the tail of the distribution of C: $P(|C| = n)$ for large n ?

Bond percolation on square lattice L^2

- Each edge of L^2 is open with probability p and closed with probability (1-p), independently of all other edges.
- Notations and definitions:
  - Product measure P_p
  - Open path = path made of open edges
  - Closed path = path made of closed edges
  - $x \leftrightarrow y$: there is an open path between vertices x and y
  - Open cluster in x: $C(x) = \{y$ such that $x \leftrightarrow y\}$. We write C for $C(0)$.
  - Box $B(n) = [-n,-n] \times [n,n]$
  - Box $B(x,n) = x + B(n)$



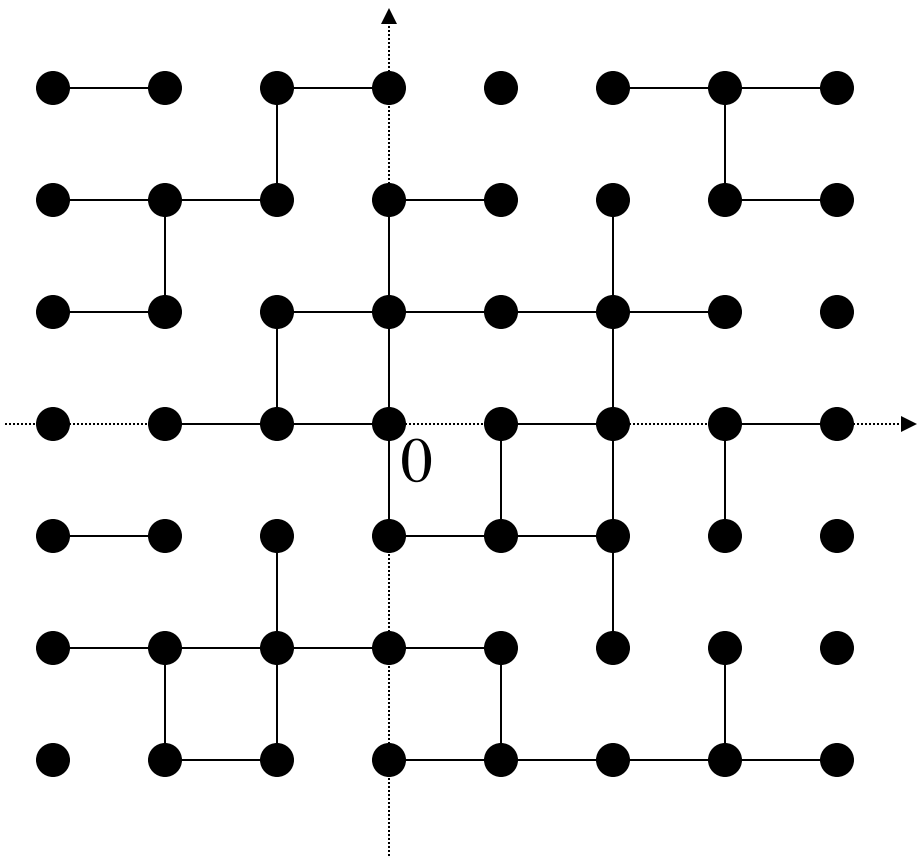
B(3)

Bond percolation on square lattice L^2

- Each edge of L^2 is open with probability p and closed with probability (1-p), independently of all other edges.
- Percolation probability:

  $\theta(p) = P_p(|C| = \infty)$
- Percolation (or critical) threshold:

  $p_c = \sup\{p : \theta(p) = 0\}$
- For L^1, $p_c = 1$ (percolation = full connectivity).
- For L^2, $1/3 \leq p_c \leq 2/3$ (percolation ≠ full connectivity).



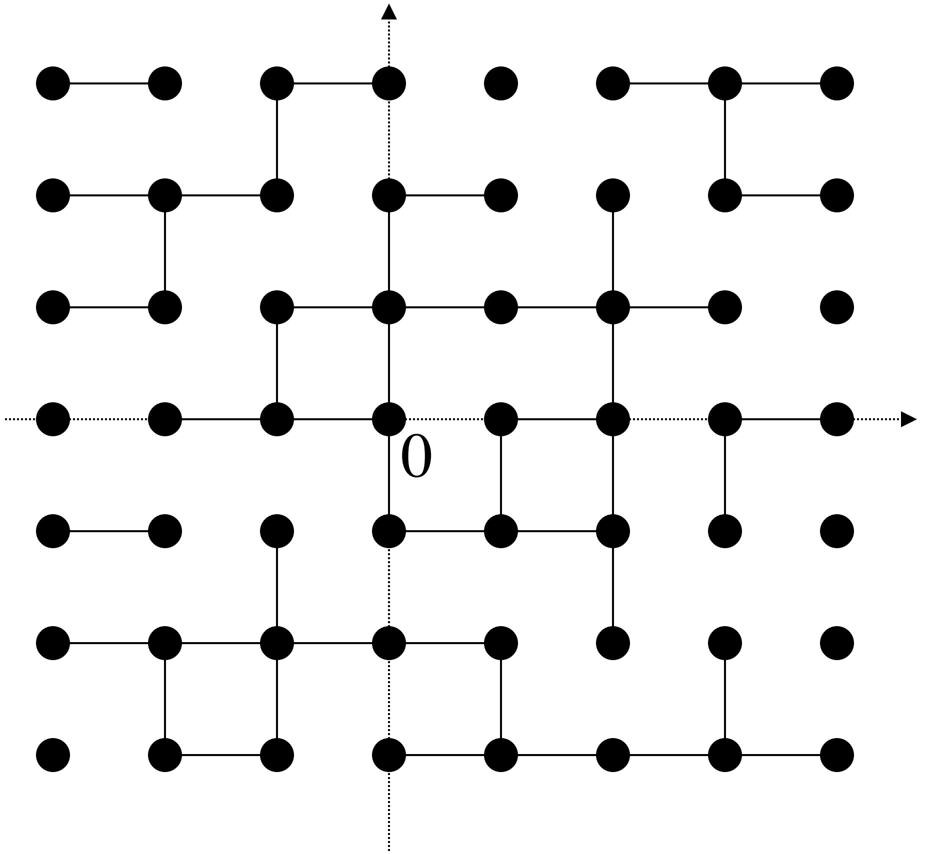
$p_c \geq 1/3$

- $\sigma(n)$ = number of self-avoiding walks of length n. $\sigma(n) \leq 4*3^{n-1}$
- $N(n)$ = number of open, self-avoiding paths of length n starting from O.
- For all integers n,

  $\theta(p) \leq P_p(N(n) \geq 1)$

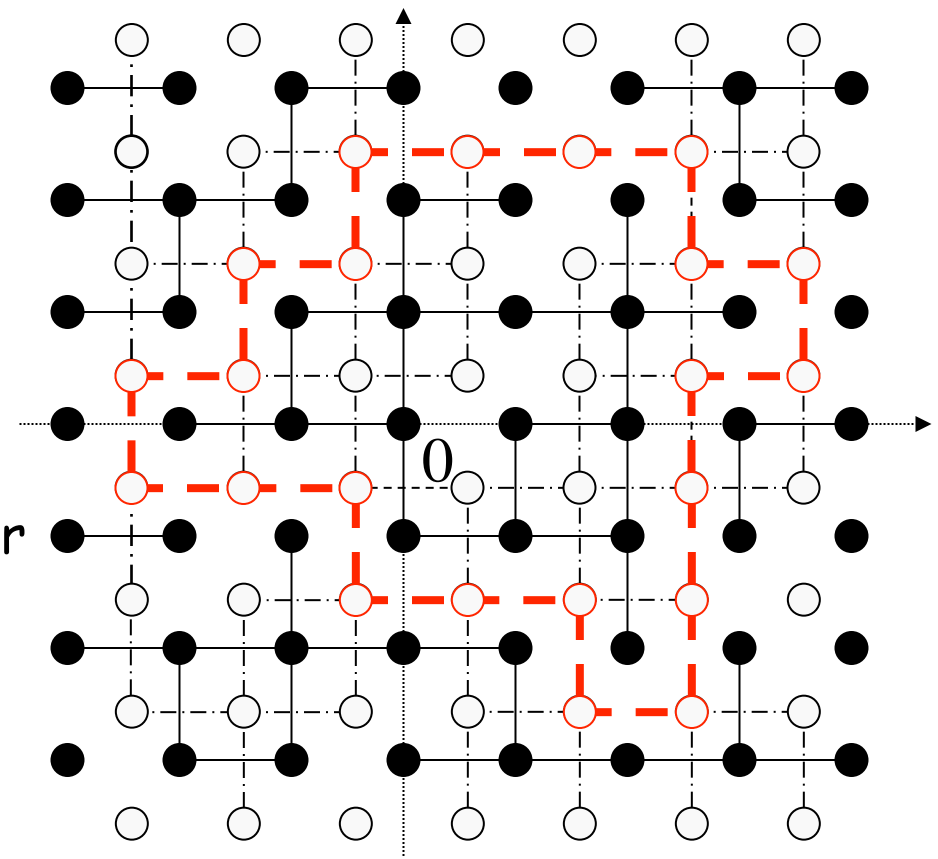
  $\leq E_p[N(n)] = \sigma(n)\, p^n$

  $\leq 4/3\, (3p)^n$
- If $p < 1/3$, $\theta(p) \leq 4/3\, (3p)^n \longrightarrow 0$ as $n \longrightarrow \infty$.
- $p_c = \sup\{p : \theta(p) = 0\} \geq 1/3$



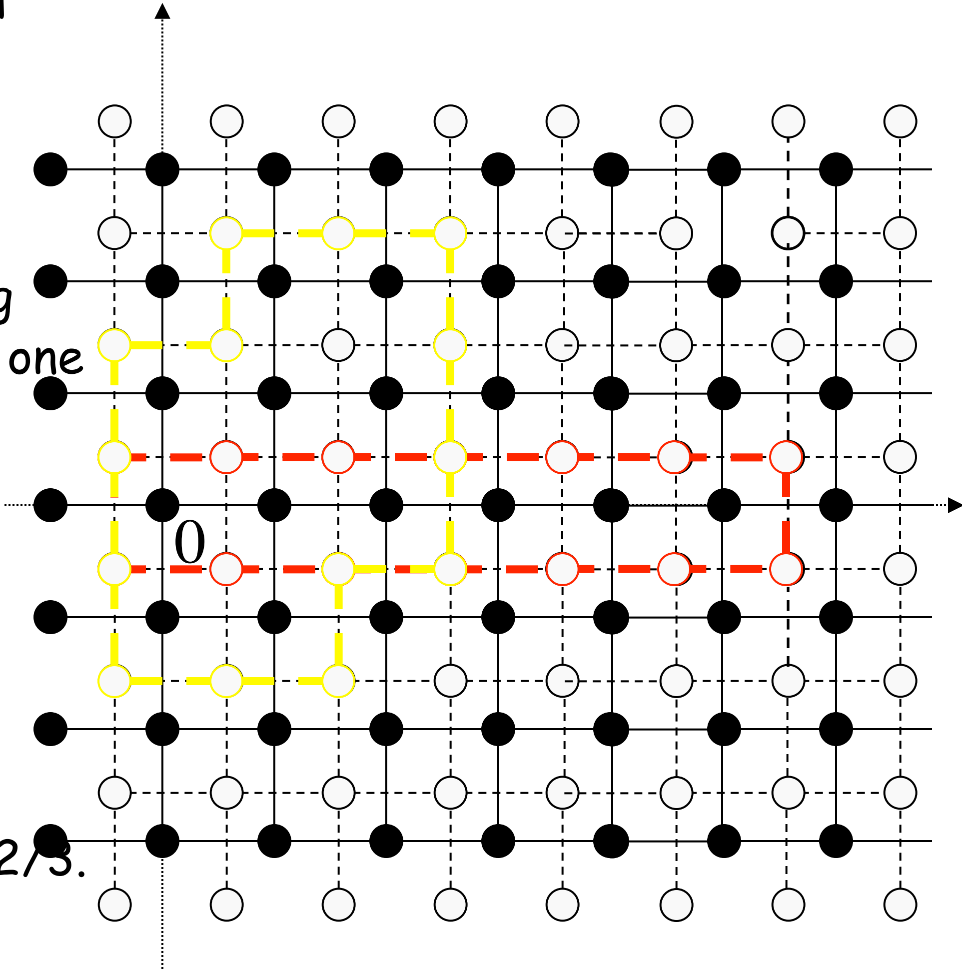
$p_c \le 2/3$: Dual technique

- Construct dual lattice L_d of L^2.
- Pick integer m, and box B(m):
  - F_m = {closed circuit in L_d encircling B(m)}; $\underline{F}_m$ = F_m does not occur
  - G_m = {all edges of B(m) are open}
- $\theta(p) \ge P_p(\underline{F}_m \cap G_m) = P_p(\underline{F}_m)\, P_p(G_m)$
- $P_p(G_m) > 0$ (m finite)
- Need to compute $P_p(F_m)$
- Need first to compute the number of closed circuits of length at least 8m



$p_c \le 2/3$: Peirls' argument

- Observation: any self avoiding closed circuit of length n surrounding 0 must cross one of the n/2 edges of L_d just at the right of 0.



- Can construct the circuit a self-avoiding walk of length n-1 starting and ending at one of these n/2 edges.

-> Number of such circuits $\le (n/2)\sigma(n-1)$

- $P_p(F_m) \le \sum^{\infty}_{n=8m} P_p$ ($\exists$ closed circuit of length n)

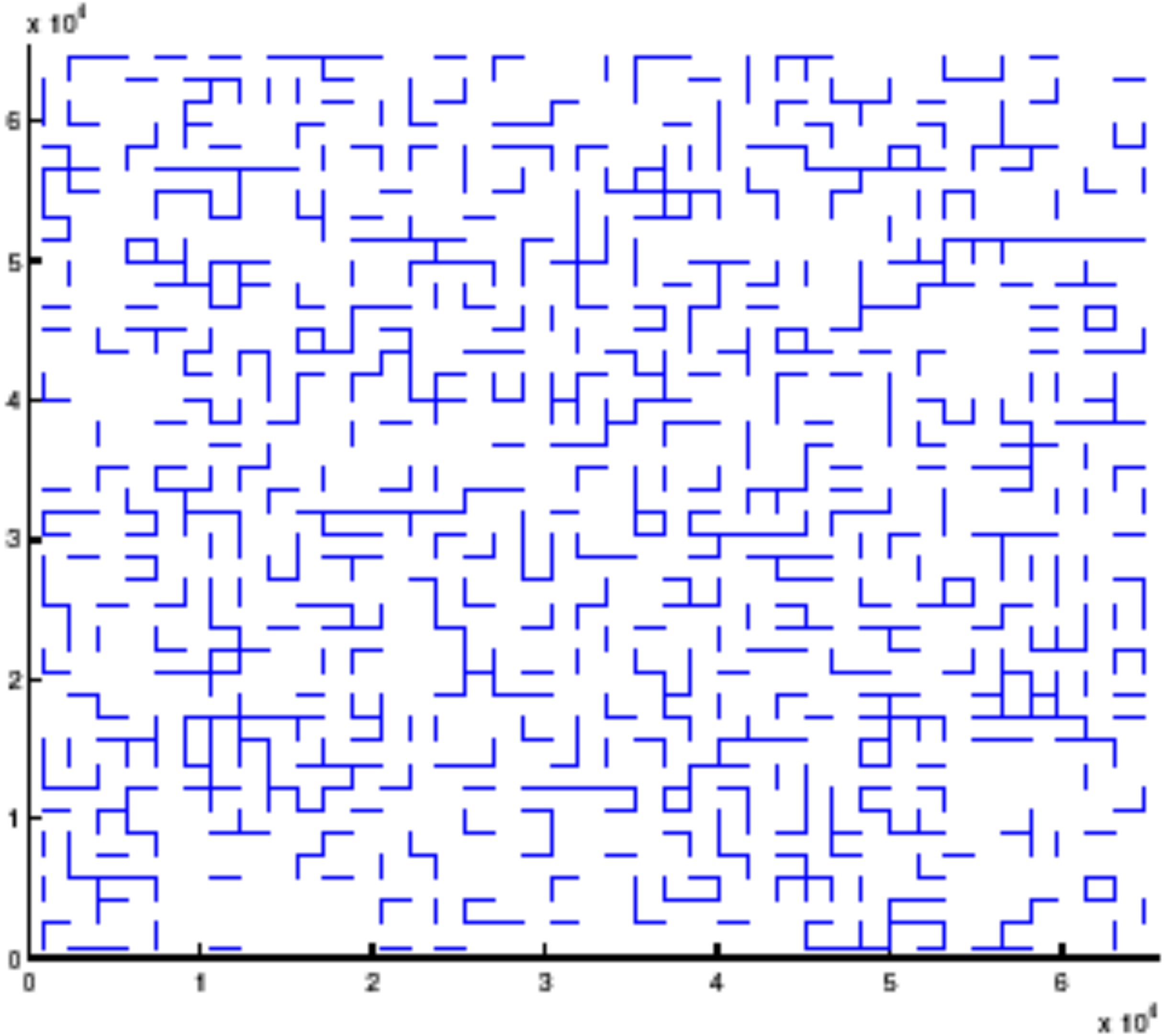
$\le \sum^{\infty}_{n=8m} (n/2)\sigma(n-1)\,(1-p)^n$

$\le (2(1-p)/3) \sum^{\infty}_{n=8m} n(3(1-p))^{n-1}$

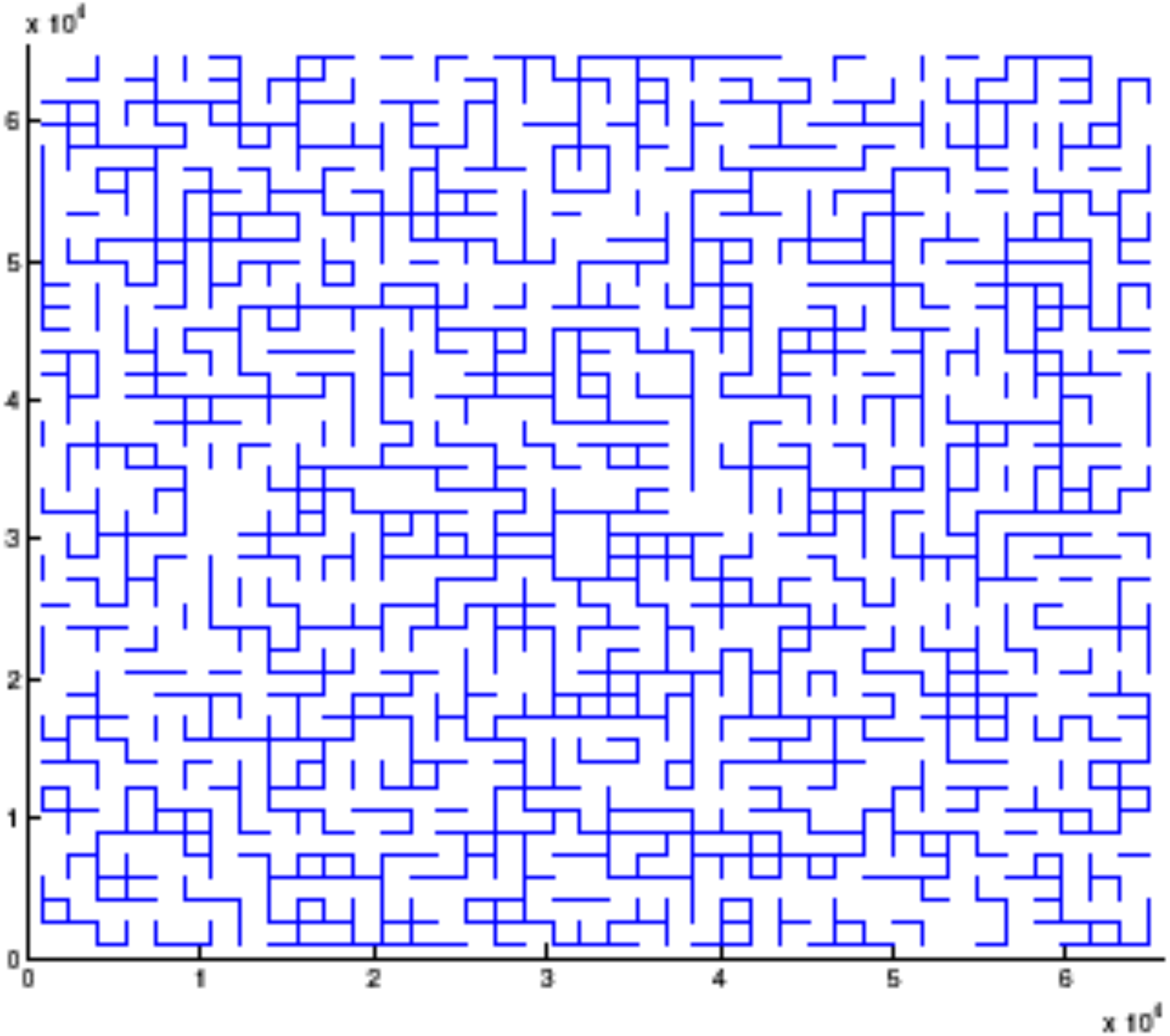
$\le 1/2$ if m is large enough and $p > 2/3$.

- $\theta(p) \ge P_p(\underline{F}_m)\, P_p(G_m) \ge P_p(G_m)/2 > 0$.
- $p_c = \sup\{p : \theta(p) = 0\} \le 2/3$.

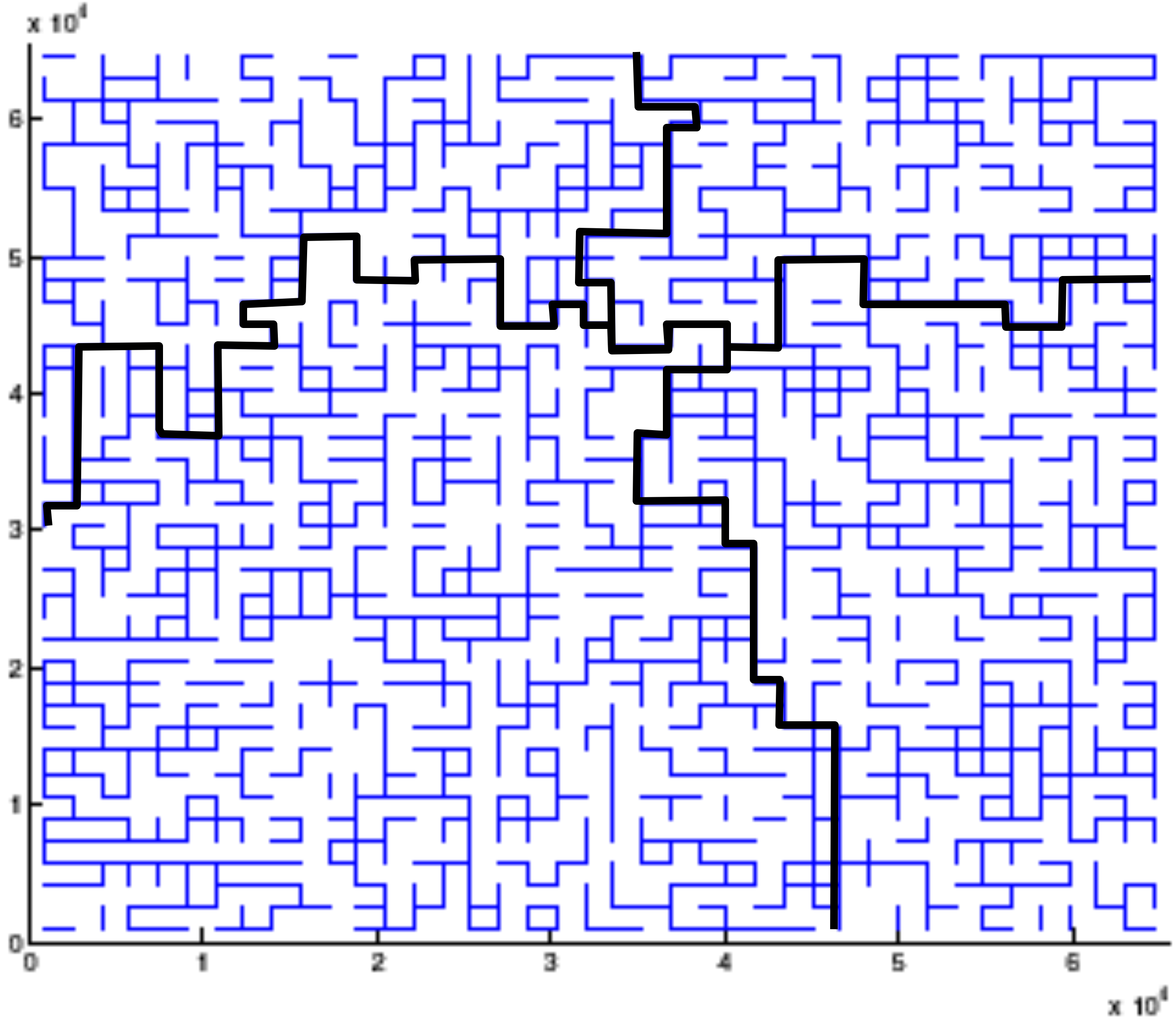
p=0.3



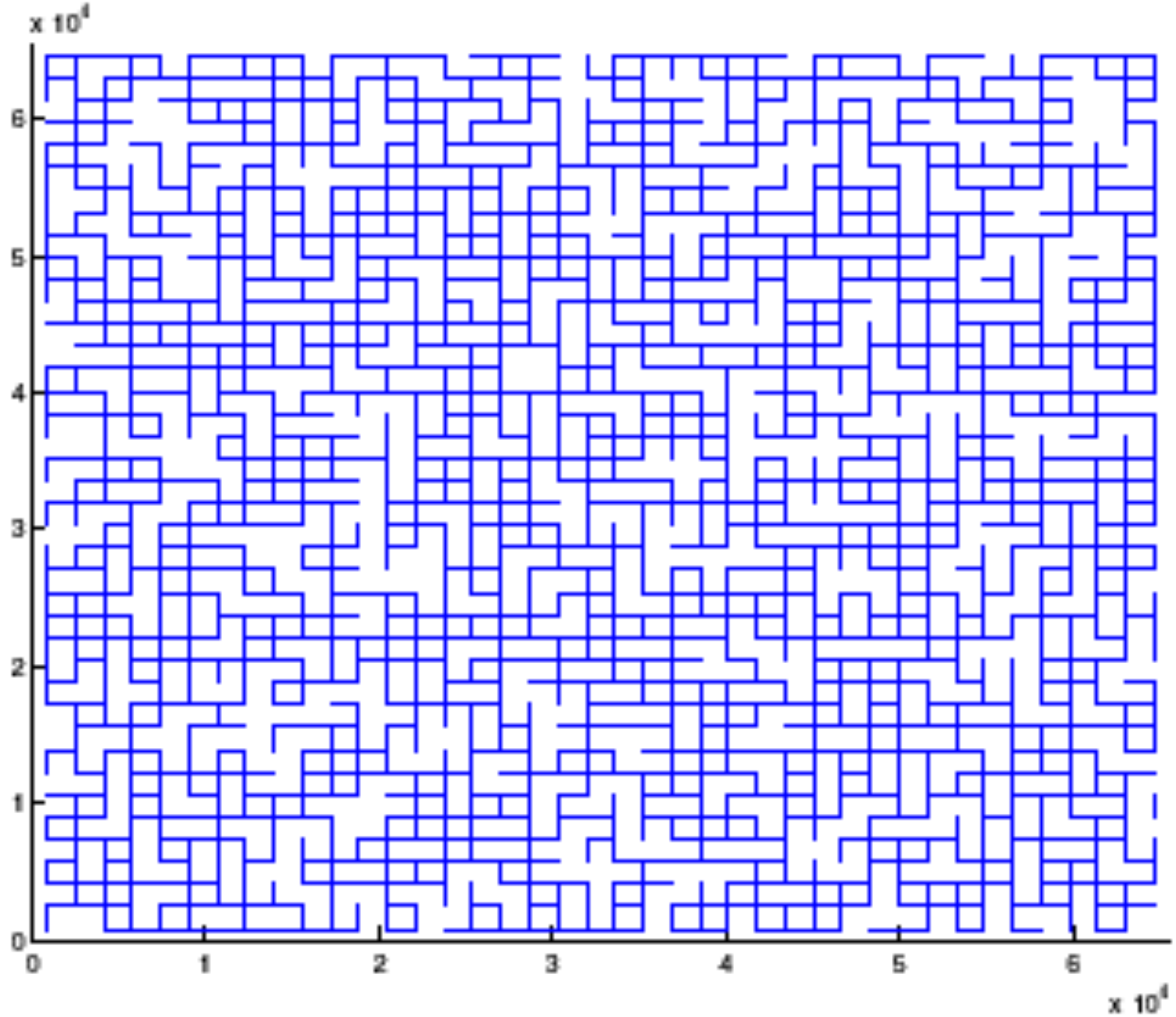
p=0.49

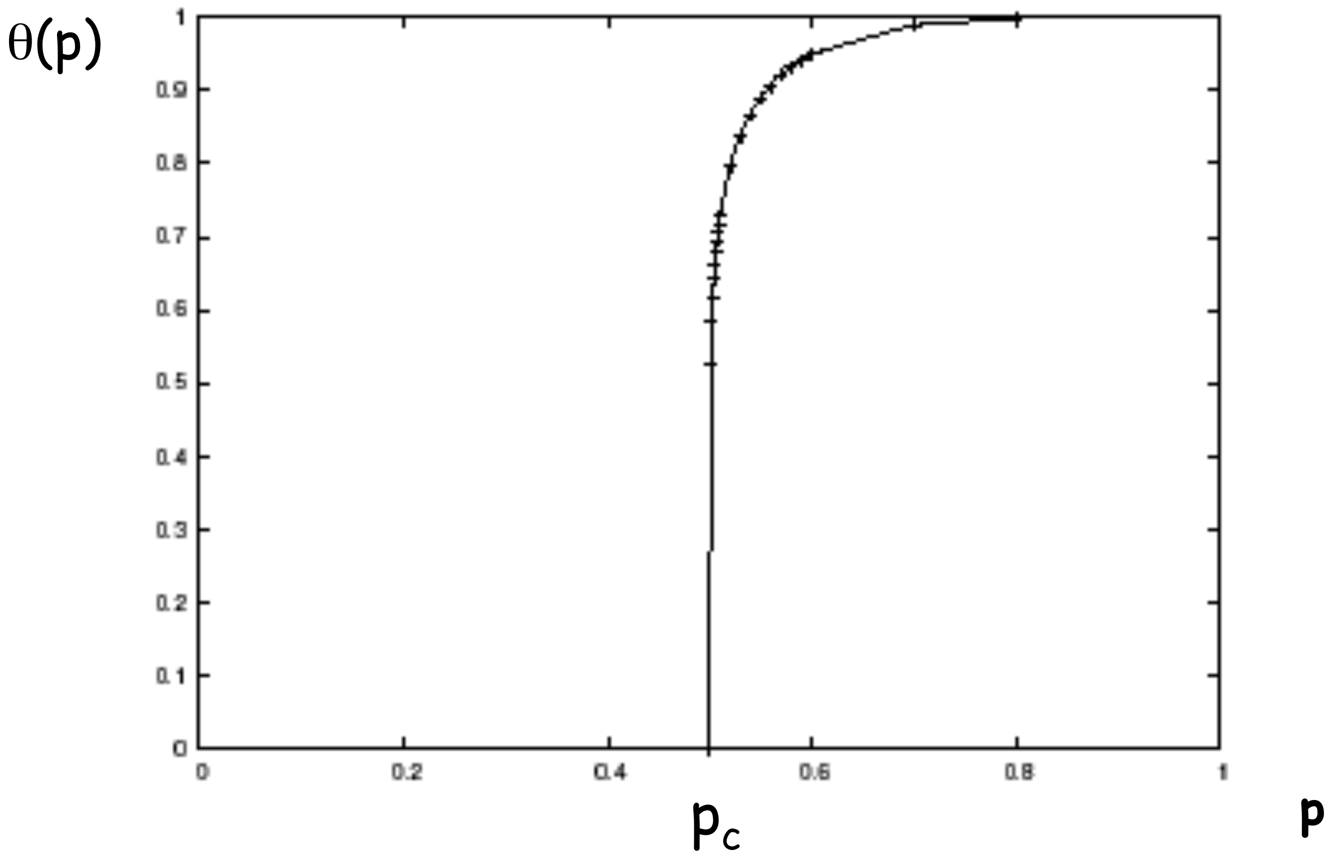


p=0.51



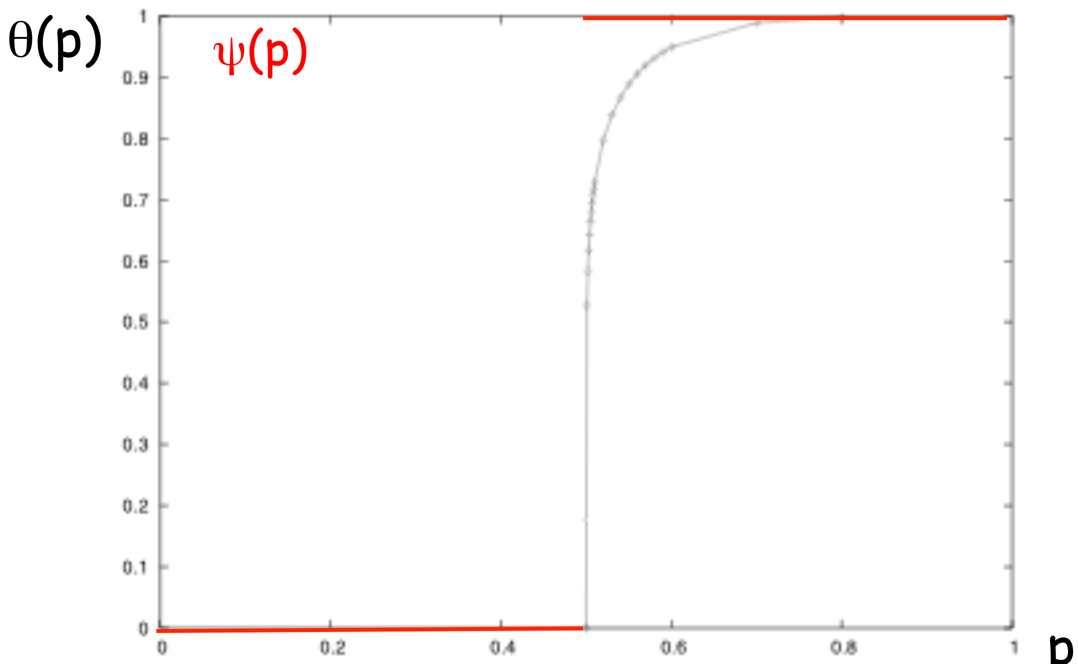
p=0.7

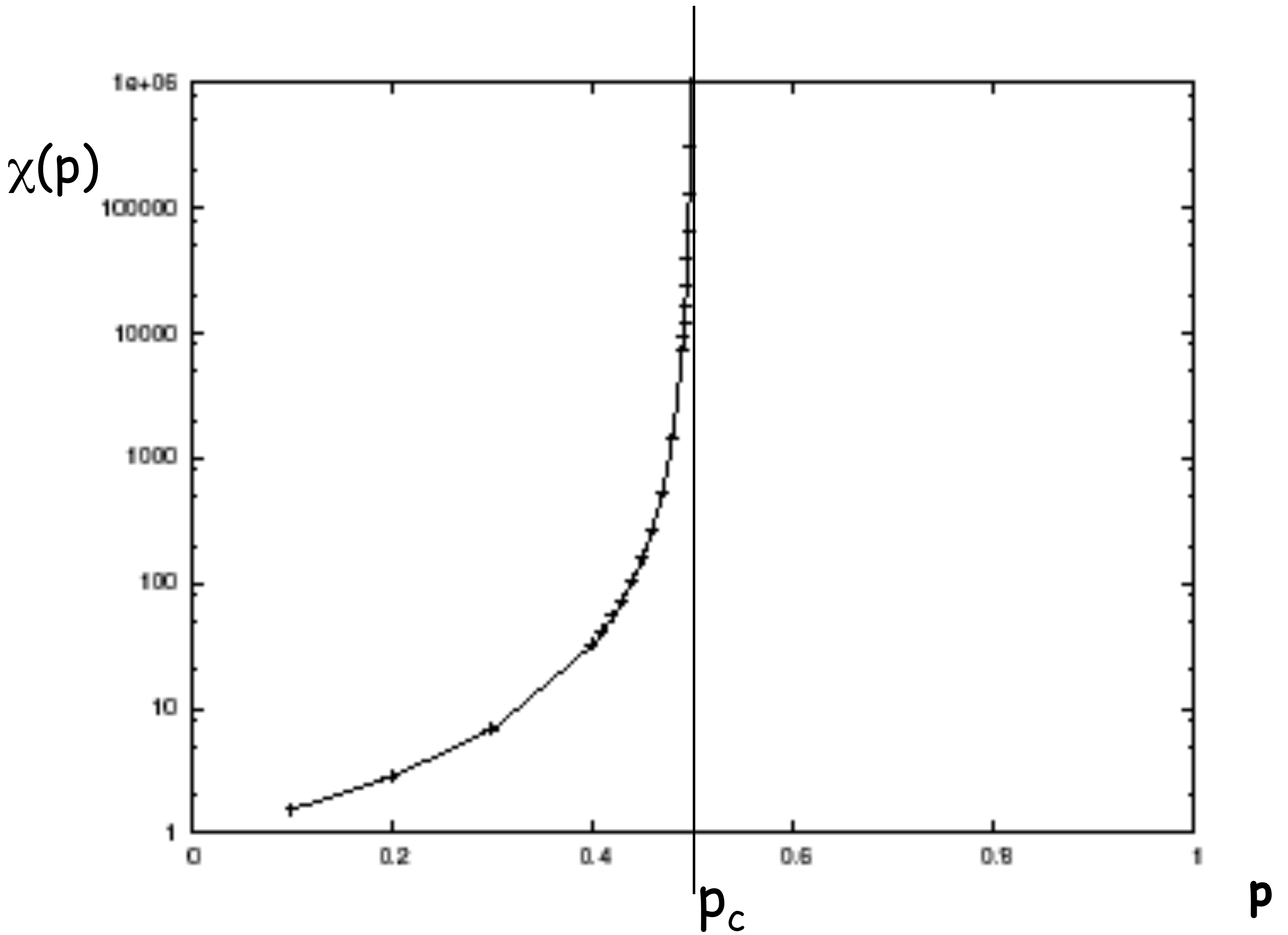



θ(p)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
p_c
p

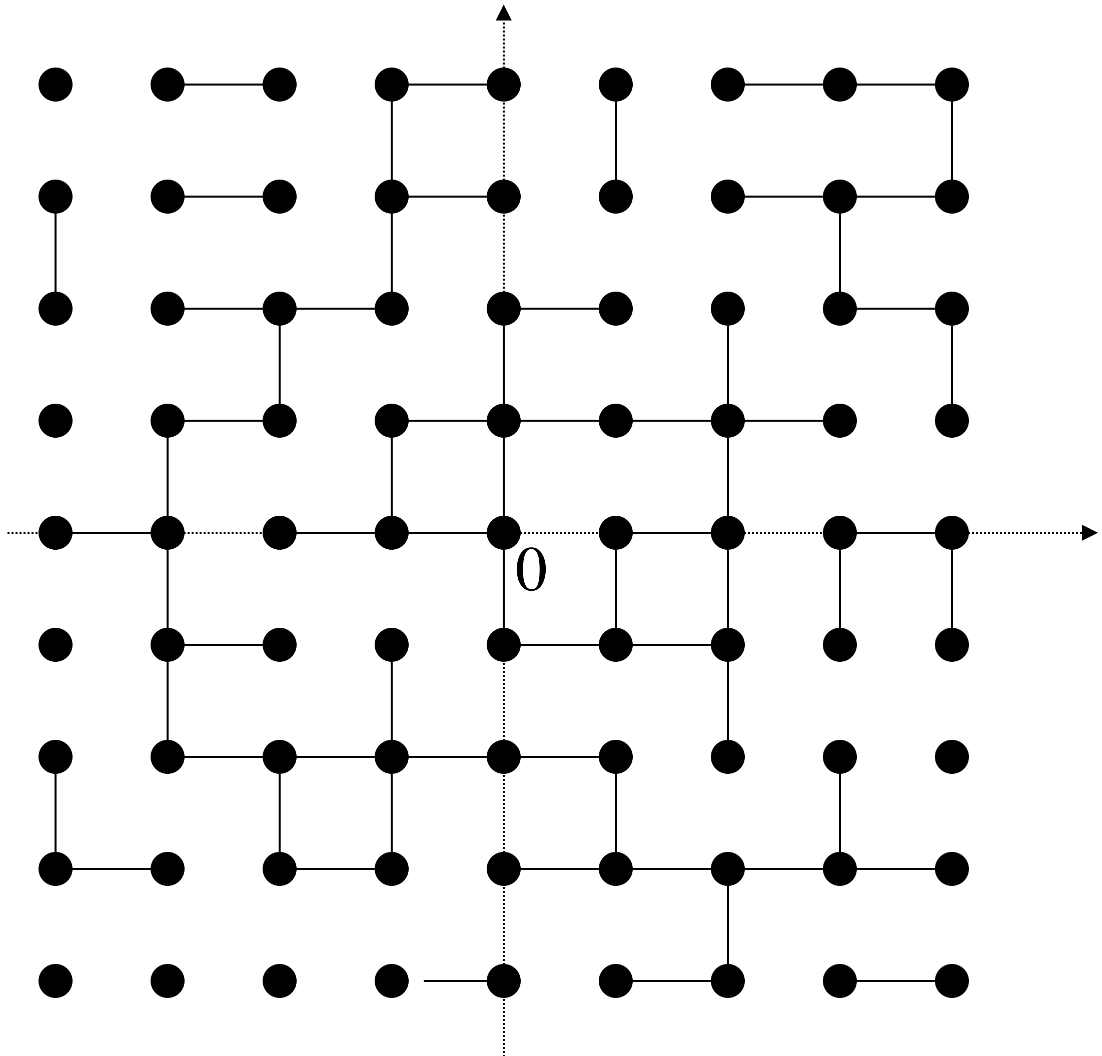
Bond percolation on square lattice L^2

- $\theta(p) = P_p(|C| = \infty) = P_p$ (a node belongs to an infinite cluster)
- $\psi(p) = P_p$ (there exists an infinite cluster)
- Existence of an infinite cluster is a tail event (does not depend on the state of any finite collection of edges).
- Kolmogorov's 0-1 law -> $\psi(p) = 0$ if $p < p_c$ and $\psi(p) = 1$ if $p > p_c$





χ(p)
1e+06
100000
10000
1000
100
10
1
0
0.2
0.4
0.6
0.8
1
p
p_c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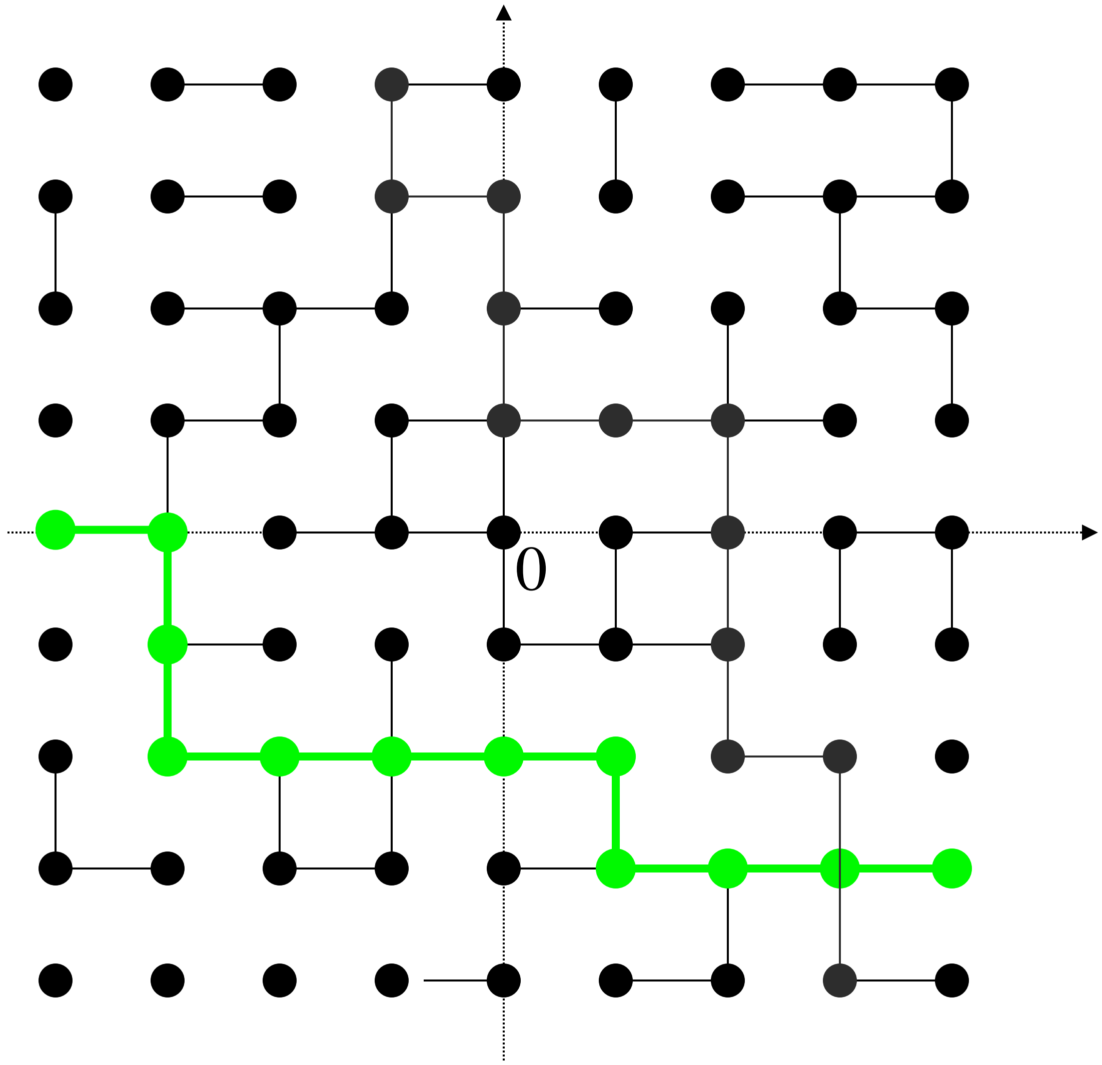
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Increasing events

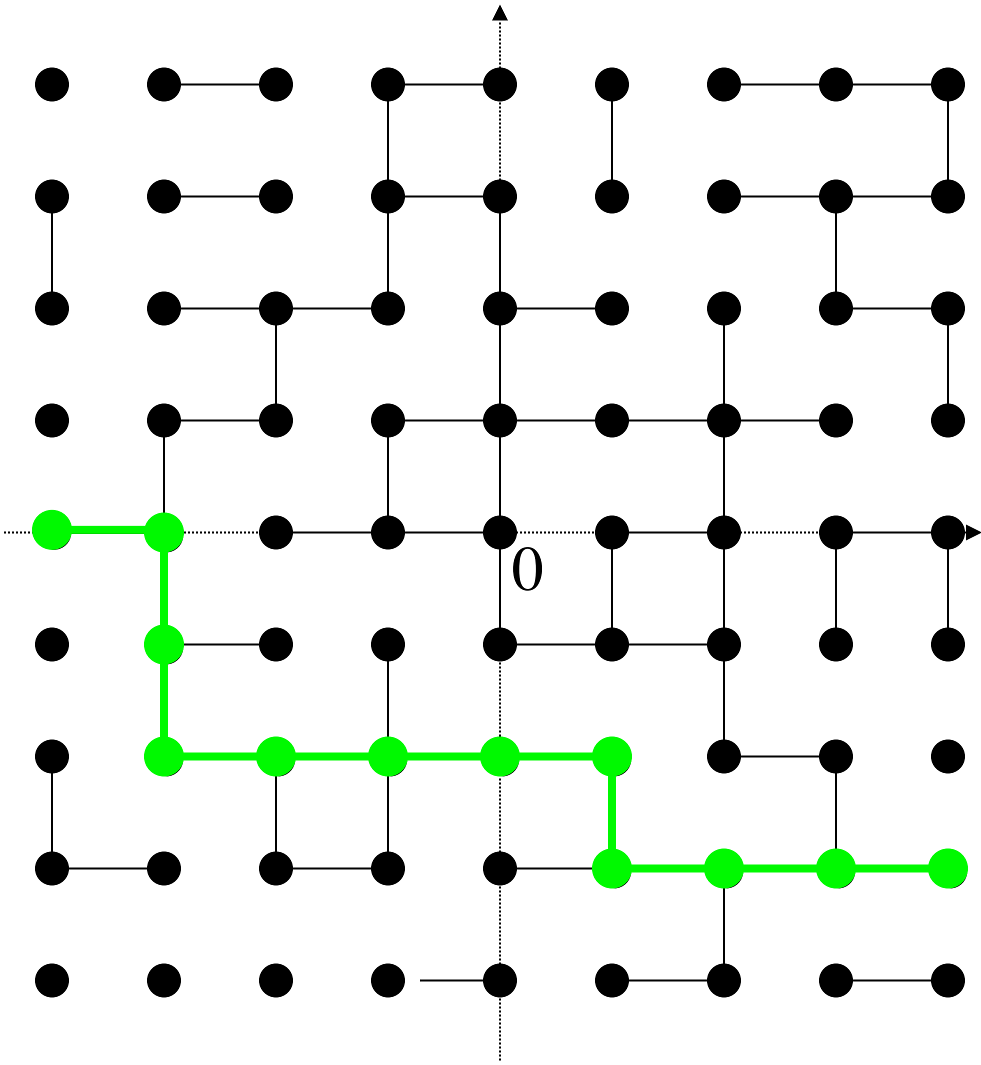
- A is an increasing event:

  $P_p(A) \le P_{p'}(A)$ if $p \le p'$.
- A is a decreasing event:

  $P_p(A) \ge P_{p'}(A)$ if $p \le p'$.
- Example:
  - A = {LR open crossing of B(n)}

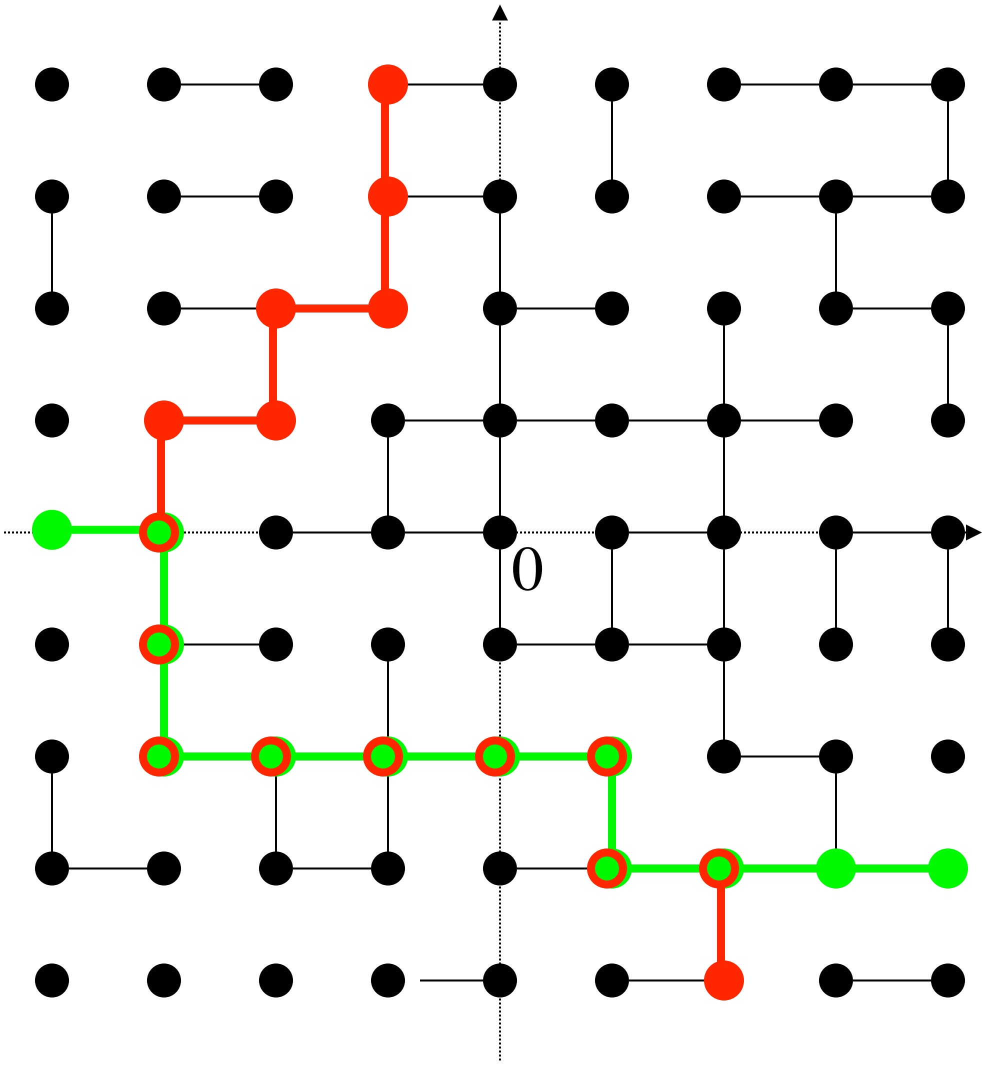
    (LR = left-right) is increasing
  - A = {TB closed path in B(n)}

    (TB = top-bottom) is decreasing



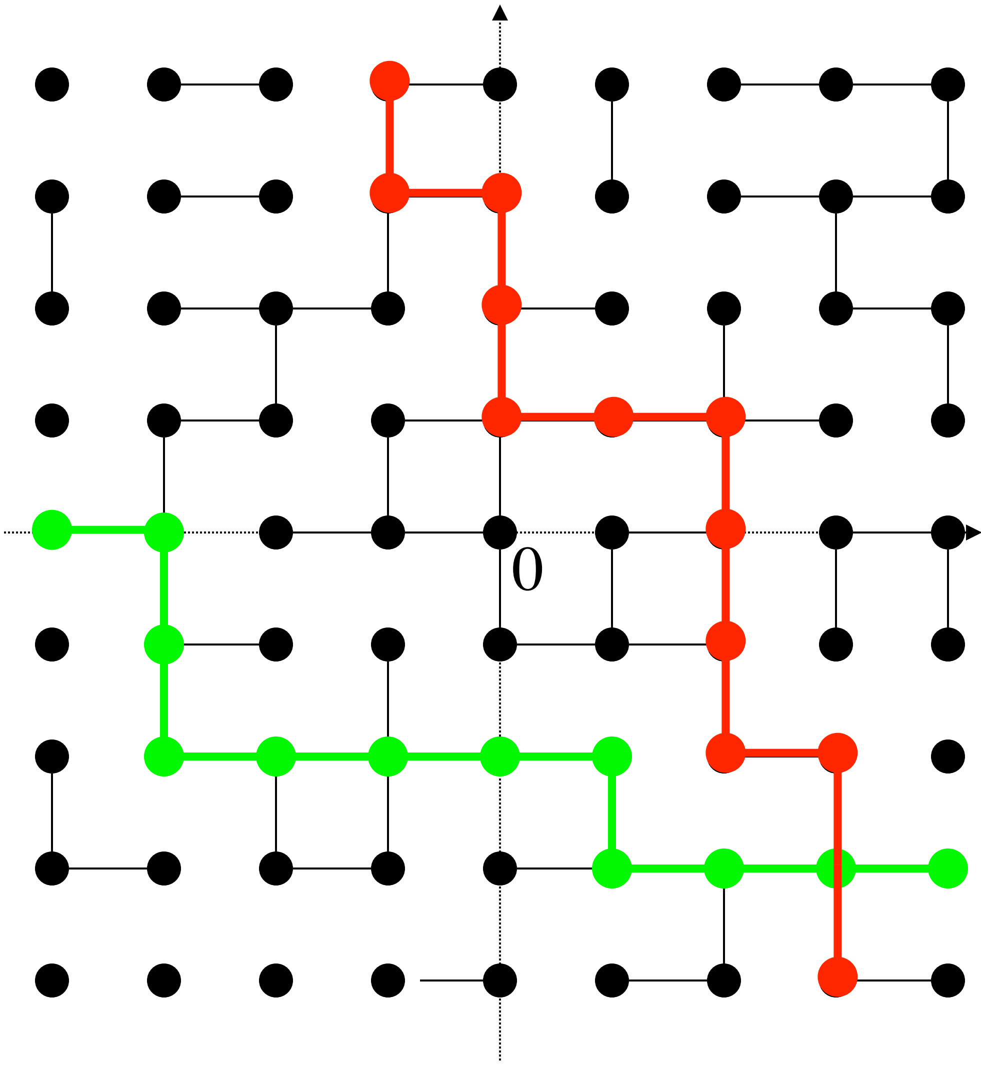
Harris' inequality (FKG inequality)

- Let A and B be two increasing (or decreasing) events
- $P_p(A \cap B) \geq P_p(A)\, P_p(B)$
- Example:
  - A = {LR open path in B(n)}
  - B = {TB open path in B(n)}



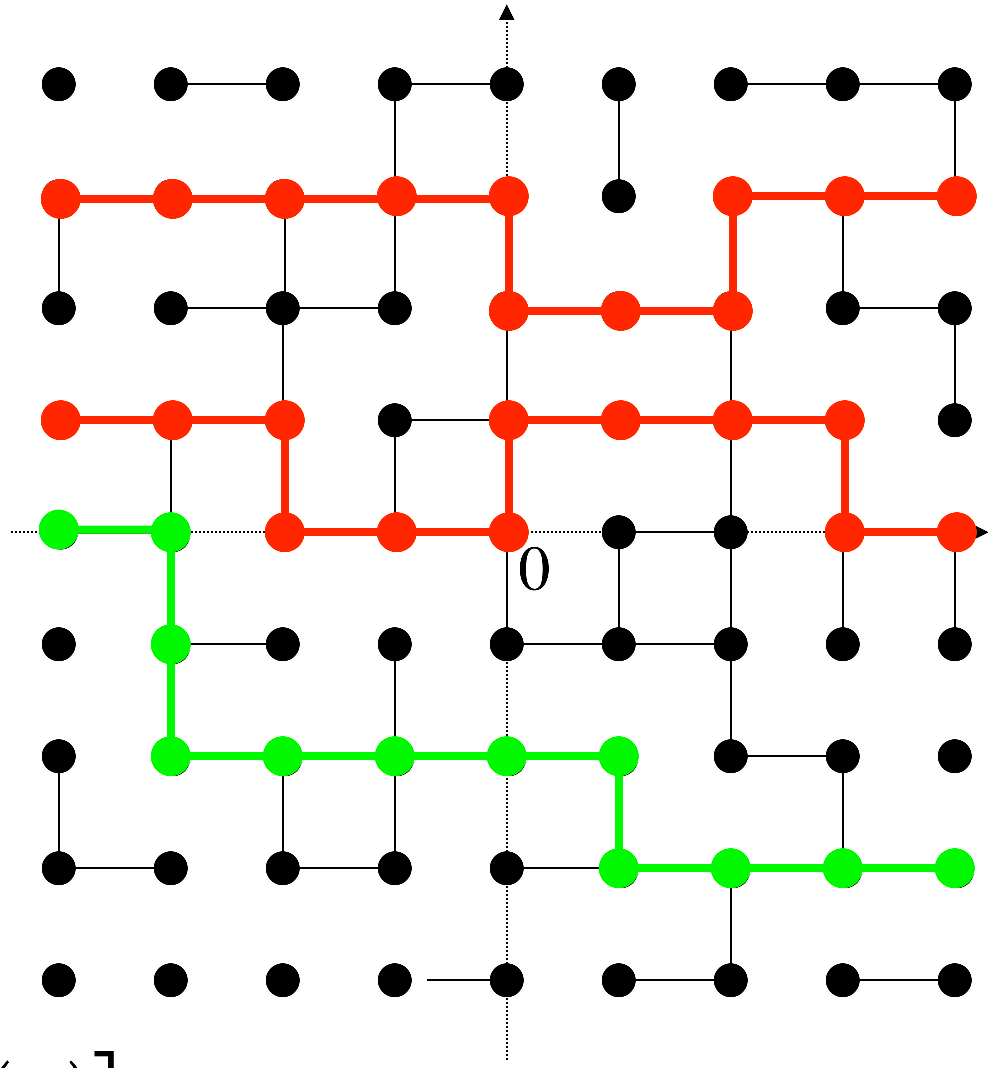
BK inequality

- Denote by $A \circ B$ the joint occurrence of A and B on disjoint sets of edges.
- Let A and B be two increasing (or decreasing) events
- $P_p(A \circ B) \leq P_p(A)\, P_p(B)$
- Example:
  - A = {LR open path in B(n)}
  - B = {TB open path in B(n)}

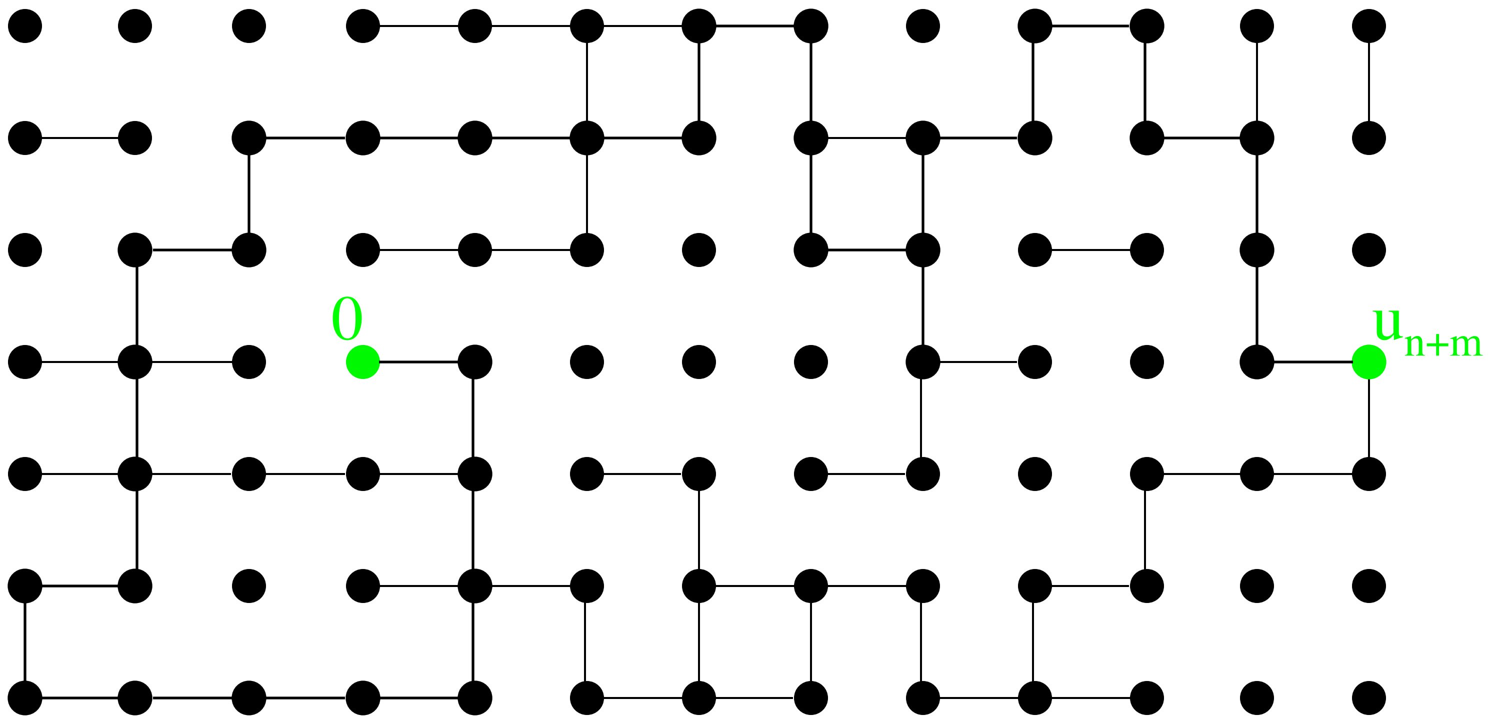


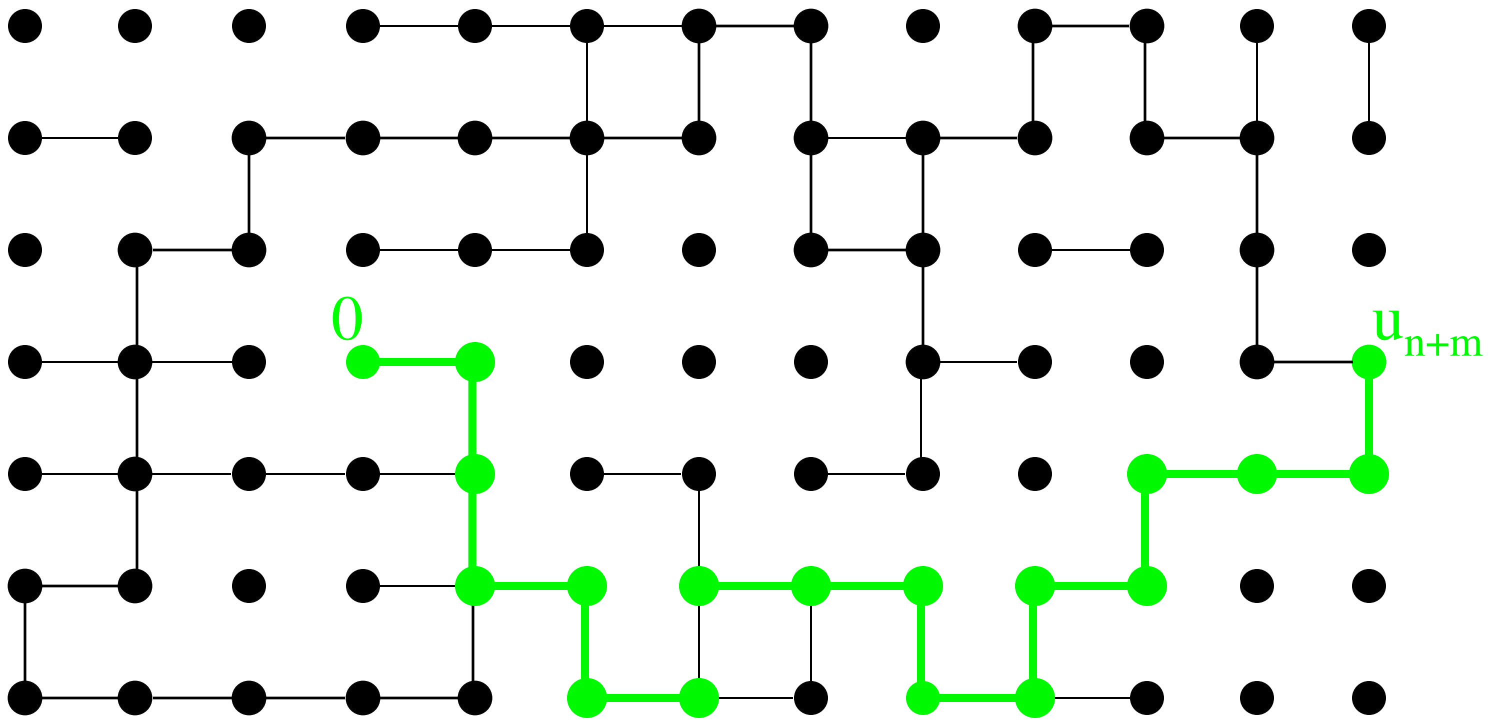
Other useful inequalities

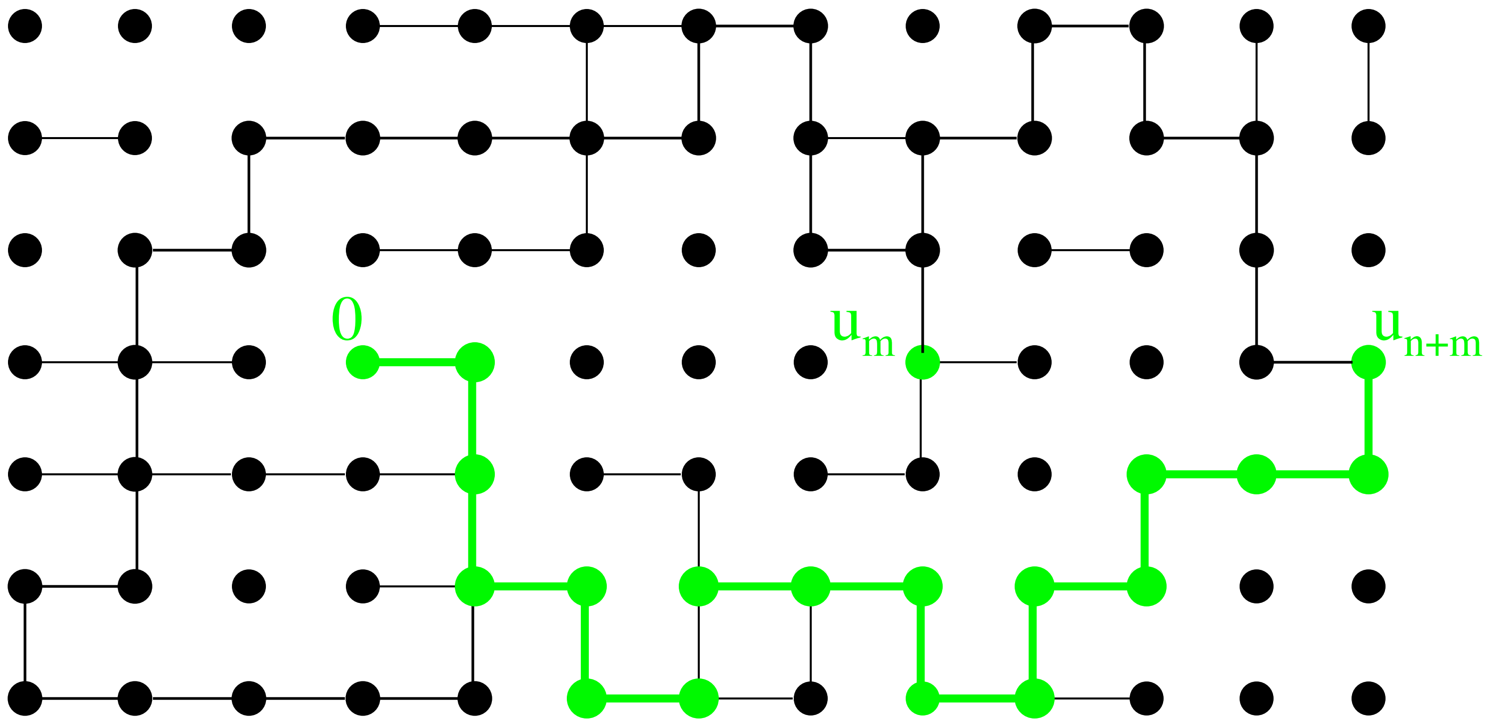
- From reliability theory (Section 2.5, 2.6 in Grimmett), for p > p'.
- Let
  - A be an increasing event
  - $I_r(A)$ be the interior of A = set of configurations in A which are still in A if we perturb up to r-1 edges.
- Example, with r = 3.
  - A = {LR open path in B(n)}
  - $I_r(A)$ = {r edge-disjoint LR open paths in B(n)}
- Theorem (Grimmett 1981):



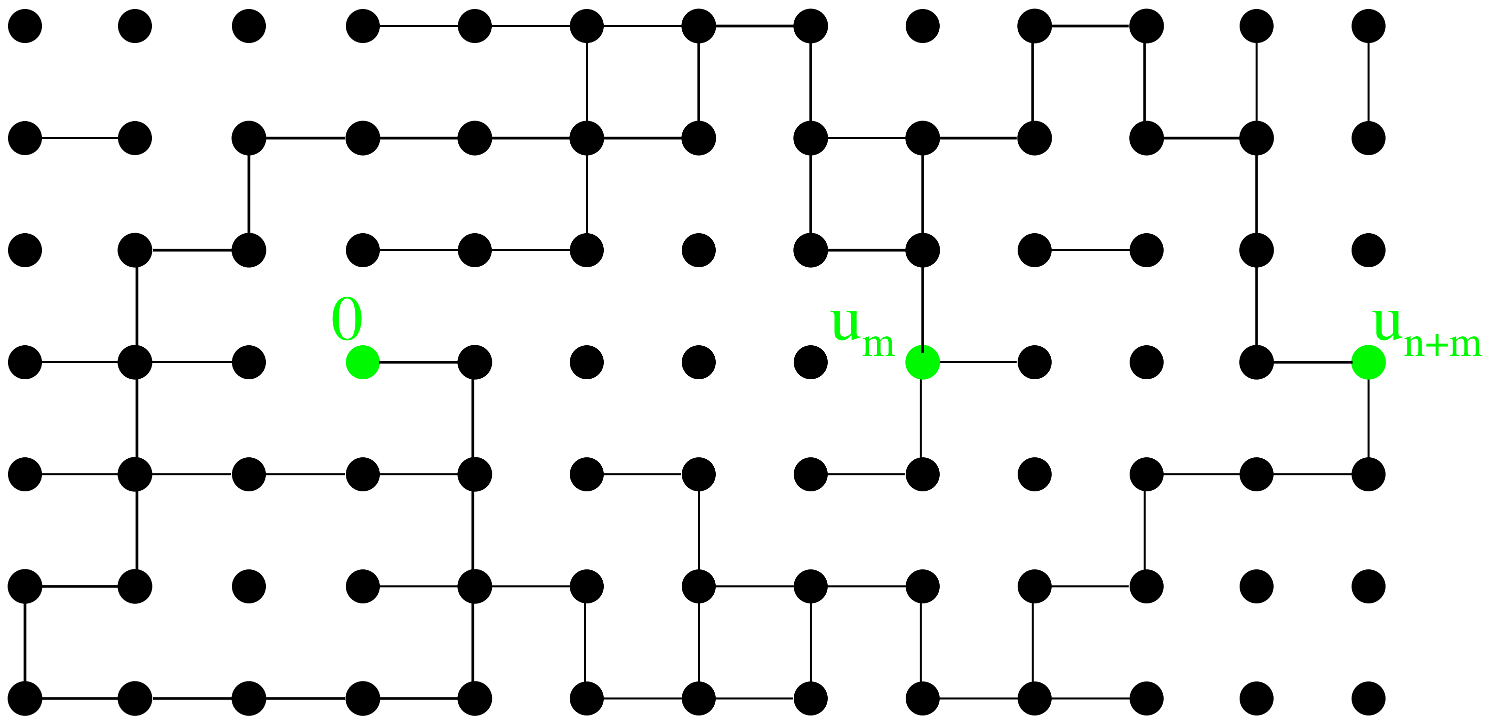
$$1 - P_p\left(I_r(A)\right) \le \left(\frac{p}{p-p'}\right)^r \left[1 - P_{p'}(A)\right]$$


0
u_{n+m}

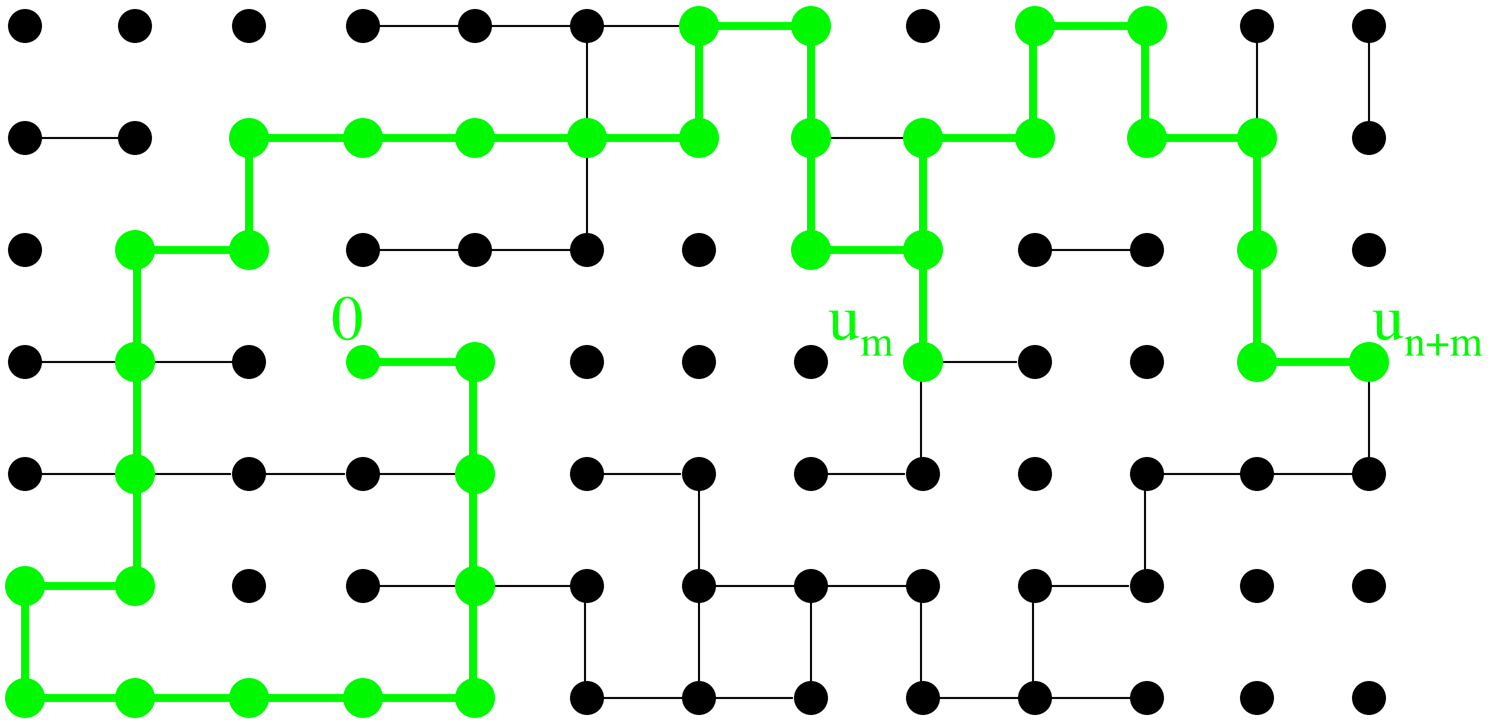

0
u_{n+m}



0
u_m
u_{n+m}



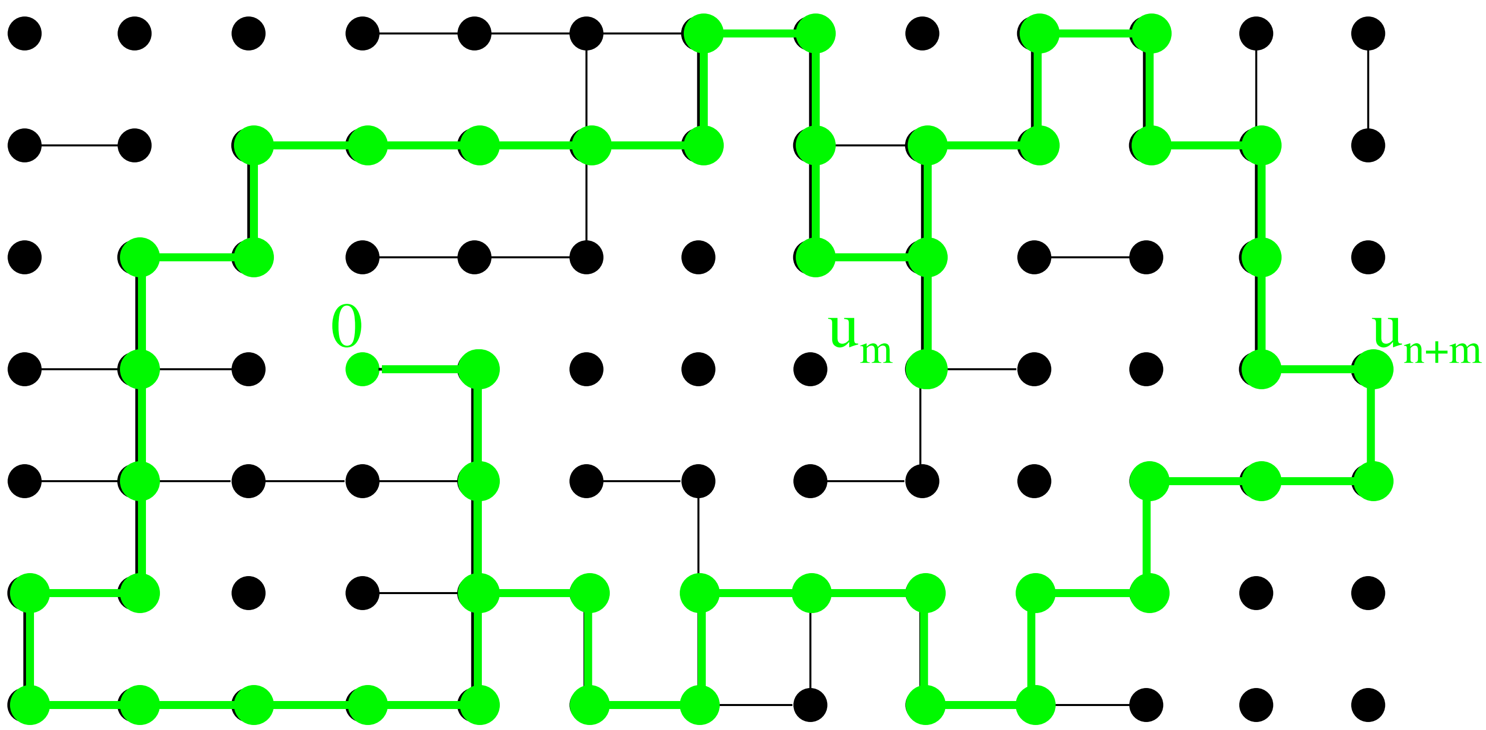
0
u_m
u_{n+m}



0
u_m
u_{n+m}

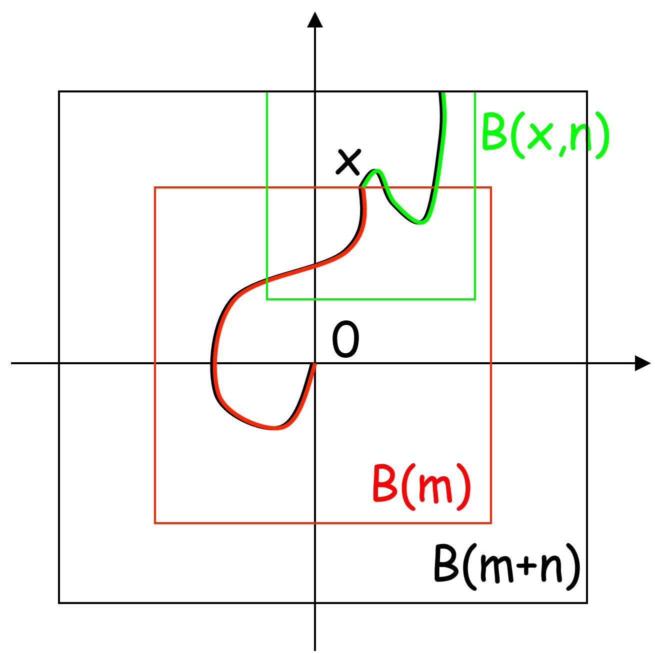
Subcritical phase

- Connectivity function $\tau(n) = P_p(0 \leftrightarrow u_n)$
- $\{0 \leftrightarrow u_m\} \cap \{u_m \leftrightarrow u_{n+m}\} \subseteq \{0 \leftrightarrow u_{n+m}\}$
- Applying FKG: $\tau(m+n) = P_p(0 \leftrightarrow u_{n+m}) \geq P_p(\{0 \leftrightarrow u_m\} \cap \{u_m \leftrightarrow u_{n+m}\})$
  $\geq P_p(0 \leftrightarrow u_m)P_p(u_m \leftrightarrow u_{n+m}) = P_p(0 \leftrightarrow u_m)P_p(0 \leftrightarrow u_n) = \tau(m)\tau(n)$
- Let $x_n = -\log(\tau(n)) \Rightarrow x_{m+n} \leq x_m + x_n$
- Sub-additive lemma: $\lim_{n\to\infty} \{x_n/n\} = x^*(p) \Rightarrow \lim_{n\to\infty} \{-1/n \log \tau(n)\} = x^*(p)$



Subcritical phase

- Let $\partial B(n)$ be the boundary of $B(n)$
- Radius function $\beta(n) = P_p(0 \leftrightarrow \partial B(n))$
- $\{0 \leftrightarrow \partial B(m+n)\} \subseteq \cup_{x \in \partial B(m)} \{ \{0 \leftrightarrow x\} \circ \{x \leftrightarrow x + \partial B(x,n)\} \}$
- Applying BK: $\beta(m+n) \leq \sum_{x \in \partial B(m)} P_p(0 \leftrightarrow x) P_p(x \leftrightarrow x + \partial B(x,n))$
  $= \sum_{x \in \partial B(m)} P_p(0 \leftrightarrow x) P_p(0 \leftrightarrow \partial B(n)) \leq |\partial B(m)|\, \beta(m)\, \beta(n)$
- Some algebraic manipulations to use the sub-additive lemma…
- $\lim_{n\to\infty} \{-1/n \log \beta(n)\} = \underline{x}^*(p)$ for some $\underline{x}^*(p)$



Subcritical phase

- $p < p_c$; $\theta(p) = P_p(|C| = \infty) = 0$
- $\chi(p) = E_p[|C|] < \infty$ (Proof is long, see Chap 5 of Grimmett)
- Exponential tails of cluster radius and size:
  - Connectivity function: $\tau(n) = P_p(0 \leftrightarrow u_n) \leq \exp(-n\, x^*(p))$ for some $x^*(p)$
  - Radius function: $\beta(n) = P_p(0 \leftrightarrow \partial B(n)) \leq \exp(-n\, \underline{x}^*(p))$ for some $\underline{x}^*(p)$
  - Easy to show that $x^*(p) = \underline{x}^*(p)$, can also prove that $x^*(p) > 0$.
  - $1/x^*(p)$ is called the correlation length.
  - Cluster size distribution, for $n > \chi^2(p)$: $P_p(|C| \geq n) \leq 2\exp(-n / 2\chi^2(p))$

Supercritical phase

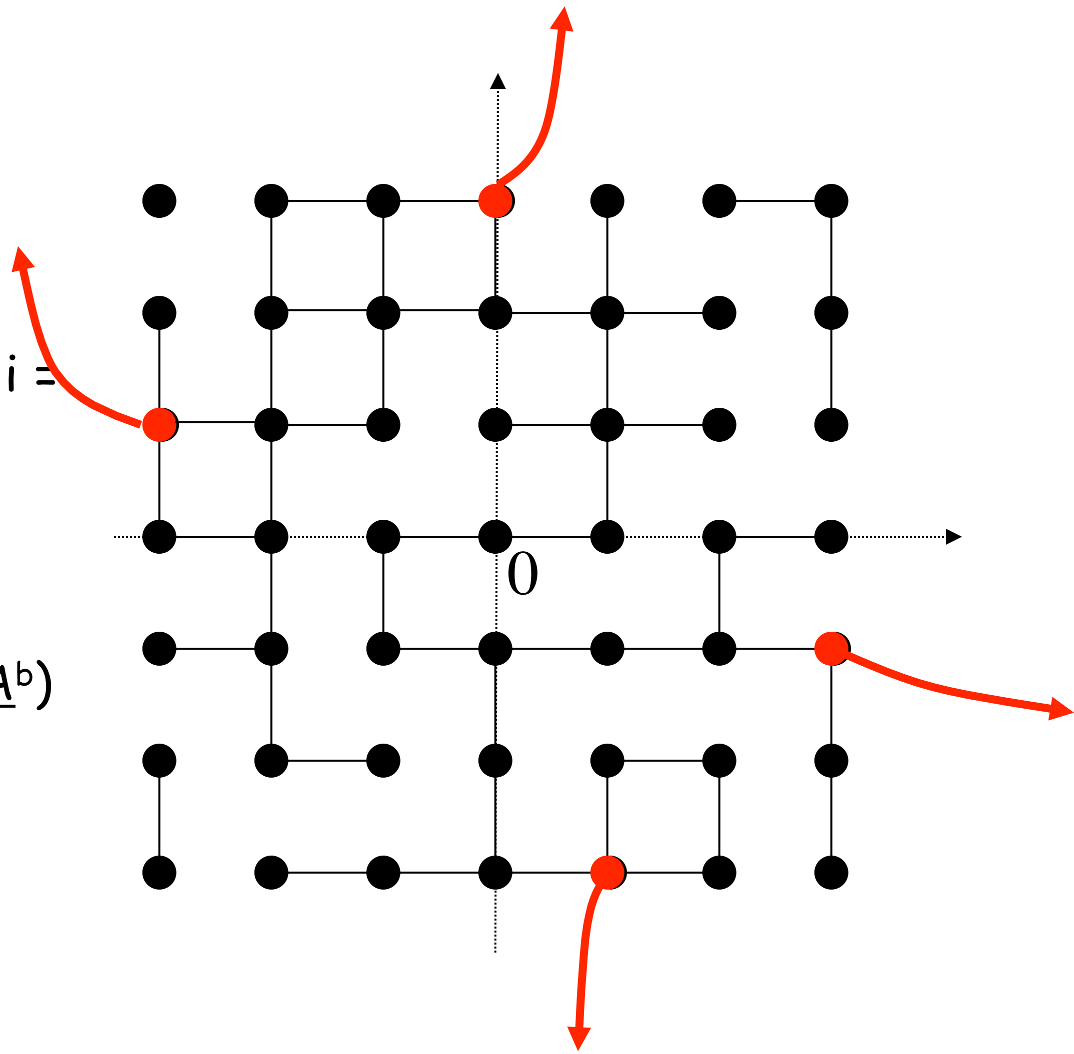
- $p > p_c$; $\theta(p) = P_p(|C| = \infty) > 0$
- We know there is a.s. at least 1 infinite cluster.
- Theorem: there is a.s. exactly 1 infinite cluster (see Chap 8 of Grimmett)

What is the value of p_c ?

- Step 1: show that $p_c \geq 1/2$. Use duality + uniqueness of infinite cluster in supercritical phase.
- Step 2: show that $p_c \leq 1/2$. Use duality + exponential decay of radius function.

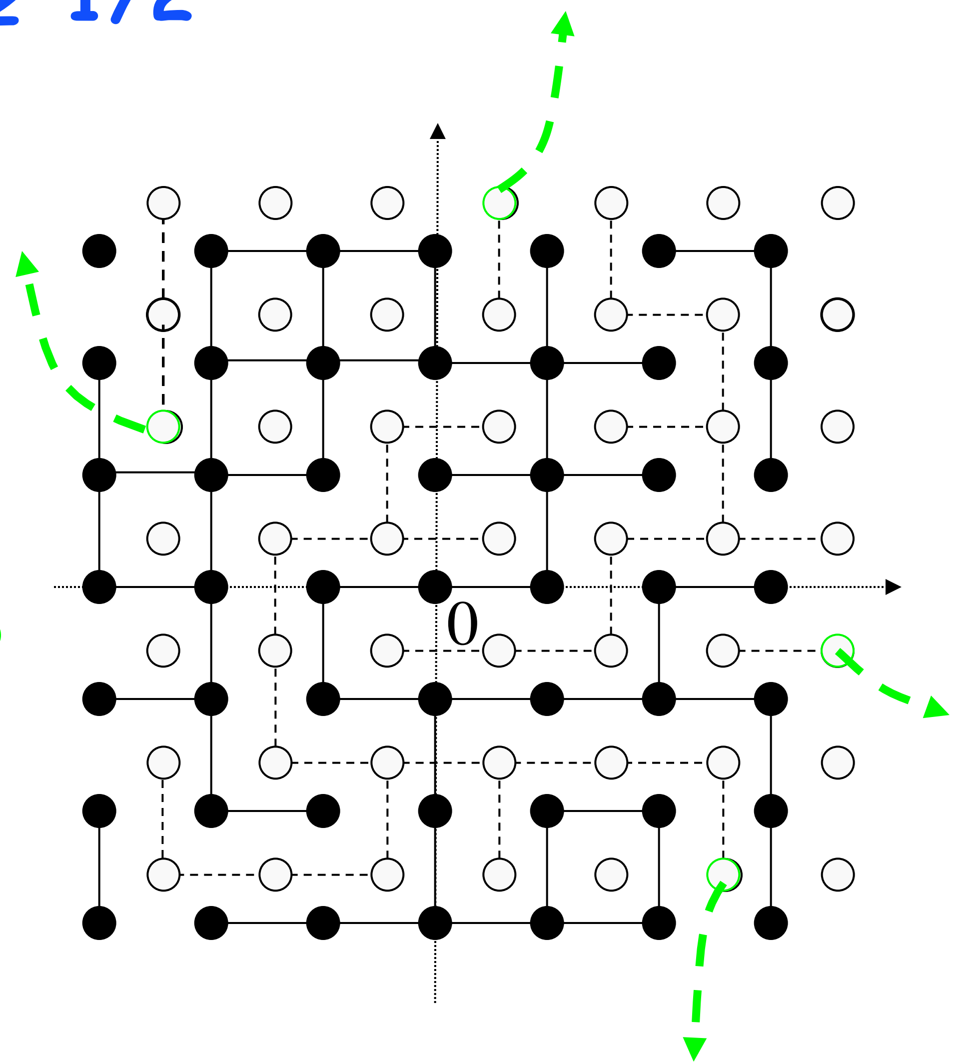
$p_c \geq 1/2$

- Zhang (1988)
- Suppose $p_c < 1/2$: $\theta(1/2) > 0$. Then there is a.s. one infinite cluster.
- Can pick integer m large enough so that $P_{1/2}(\partial B(m) \leftrightarrow \infty) > 1 - 1/8^4$
- Let A^i = {side i of B(m) is joined to the infinite cluster off B(m)} with i = r, l, t, b for resp. the right, left, top, bottom edge of B(m).
- $P_{1/2}(\partial B(m) \leftrightarrow \infty)$
  $= 1 - P_{1/2}(\underline{A}^r \cap \underline{A}^l \cap \underline{A}^t \cap \underline{A}^b)$
  $\leq 1 - P_{1/2}(\underline{A}^r)\, P_{1/2}(\underline{A}^l)\, P_{1/2}(\underline{A}^t)\, P_{1/2}(\underline{A}^b)$
  $= 1 - (P_{1/2}(\underline{A}^r))^4$
  by FKG and symmetry.
- $P_{1/2}(A^i) \geq 1 - (1 - P_{1/2}(\partial B(m) \leftrightarrow \infty))^4$
  $> 7/8$ for i = r, l, t, b.



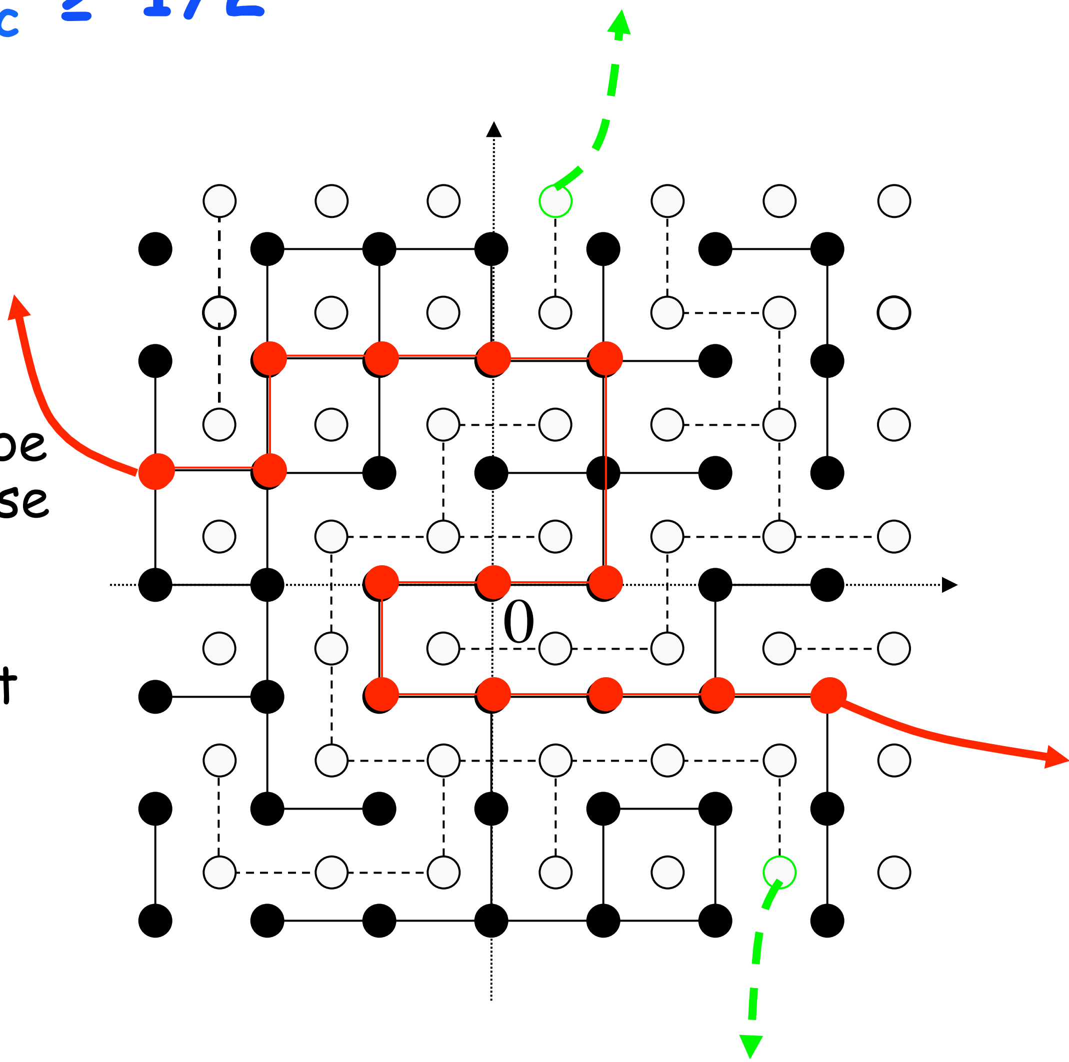
$p_c \geq 1/2$

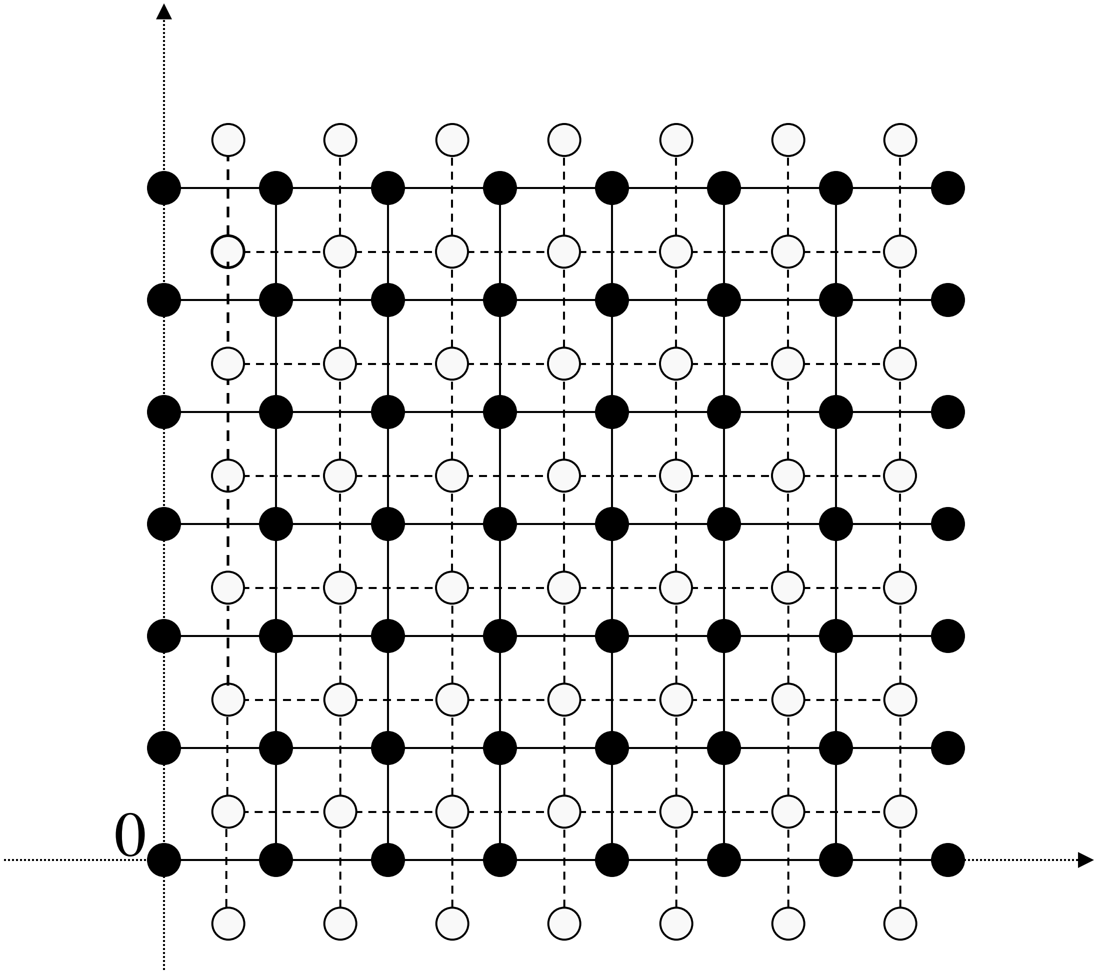
- Repeat the same with dual box.
- Each edge of L_d is closed with probability 1/2.
- If there is a.s an infinite cluster of open edges in L, there is therefore a.s an infinite cluster of closed edges in L_d
- Let A_d^i = {side i of $B_d(m)$ is joined to the infinite closed cluster off $B_d(m)$} with i = r, l, t, b for resp. the right, left, top, bottom edge of $B_d(m)$.
- $P_{1/2}(A_d^i) > 7/8$ for i = r, l, t, b.



$p_c \geq 1/2$

- Let $A = A^r \cap A^l \cap A_d^t \cap A_d^b$
- $P_{1/2}(A) =$
  $1 - P_{1/2}(\underline{A}^r \cup \underline{A}^l \cup \underline{A}_d^t \cup \underline{A}_d^b)$
  $\geq 1 - (P_{1/2}(\underline{A}^r) + P_{1/2}(\underline{A}^l)$
  $+ P_{1/2}(\underline{A}_d^t) + P_{1/2}(\underline{A}_d^b)) = 1/2.$
- If $A^r \cap A^l$ occurs, there must be an LR open path in B(m), because the open infinite cluster is unique.
- If $A_d^t \cap A_d^b$ occurs, there must be a TB closed path in B(m), because the open (closed) infinite cluster is unique.
- But then $P_{1/2}(A) = 0$, a contradiction.
- Therefore $\theta(1/2) = 0$.

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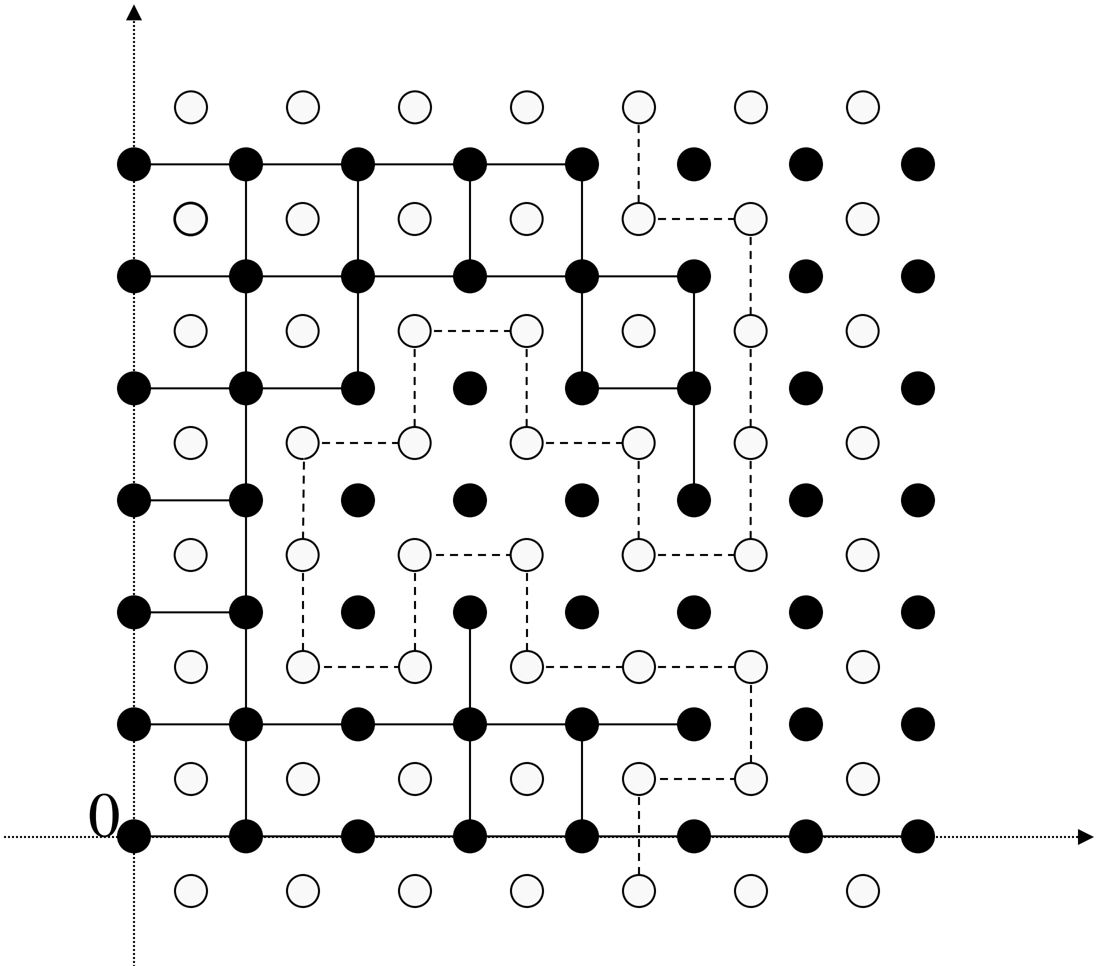
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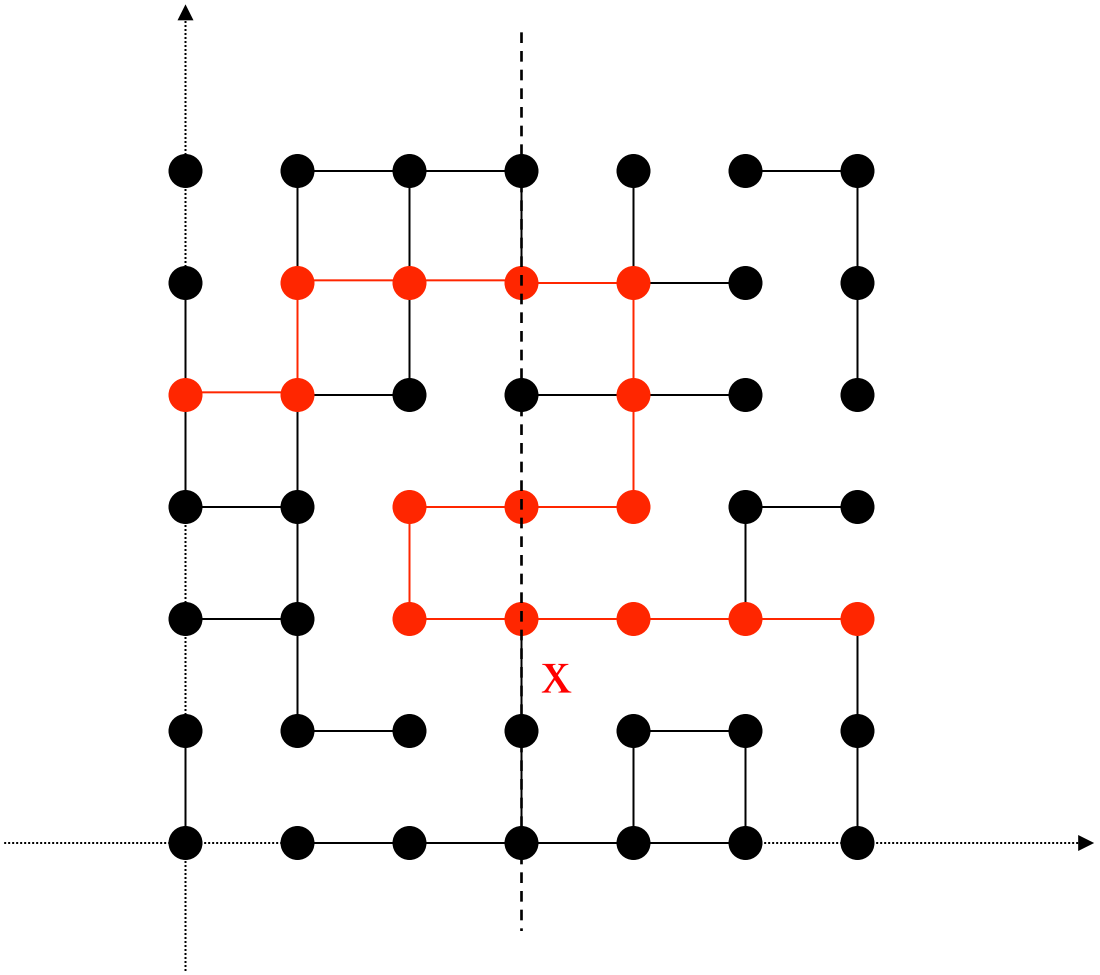
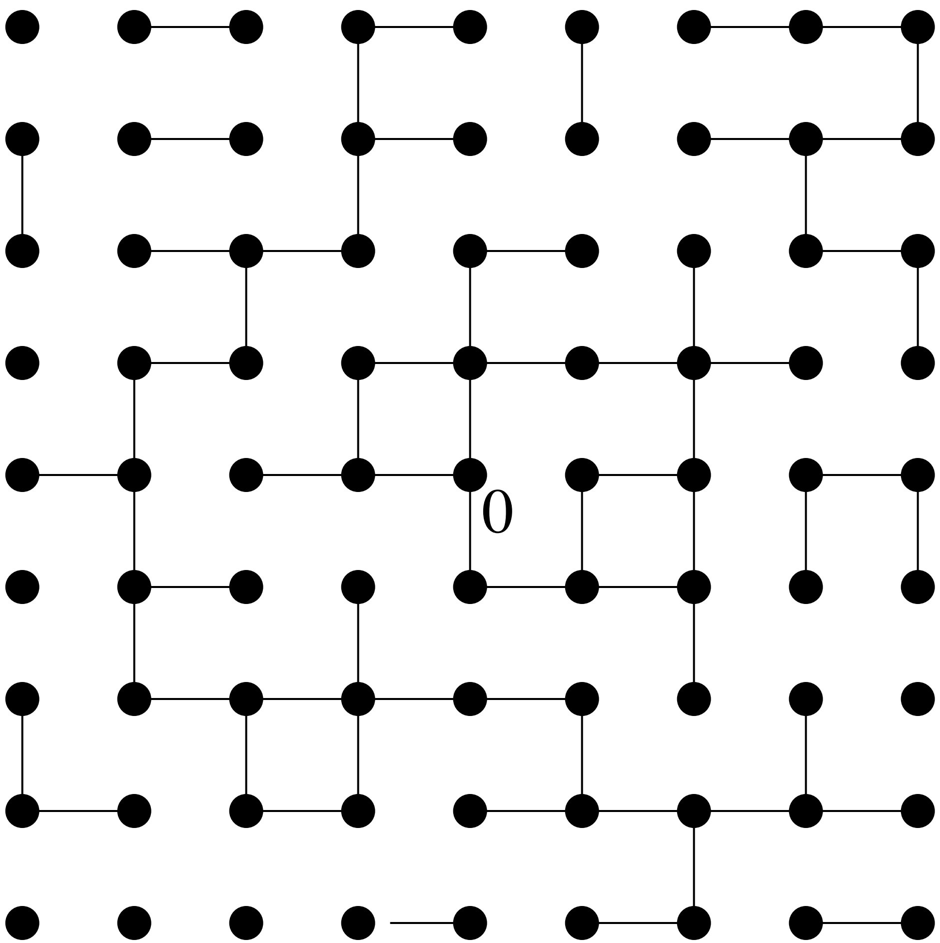

X

Lattice bond percolation



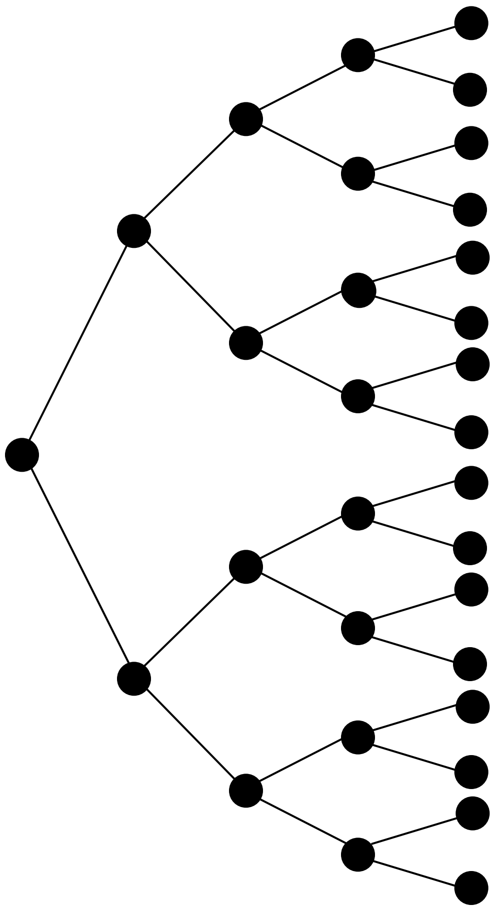
- Infinite lattice
- Each edge is « open » with probability p (i.i.d)
- Let C be the connected component containing 0.
- What is the probability

$$\theta(p) = P(|C| = \infty) ?$$

- There exists p_c such that
  - $\theta(p) = 0$ if $p < p_c$
  - $\theta(p) > 0$ if $p > p_c$

Tree bond percolation

0

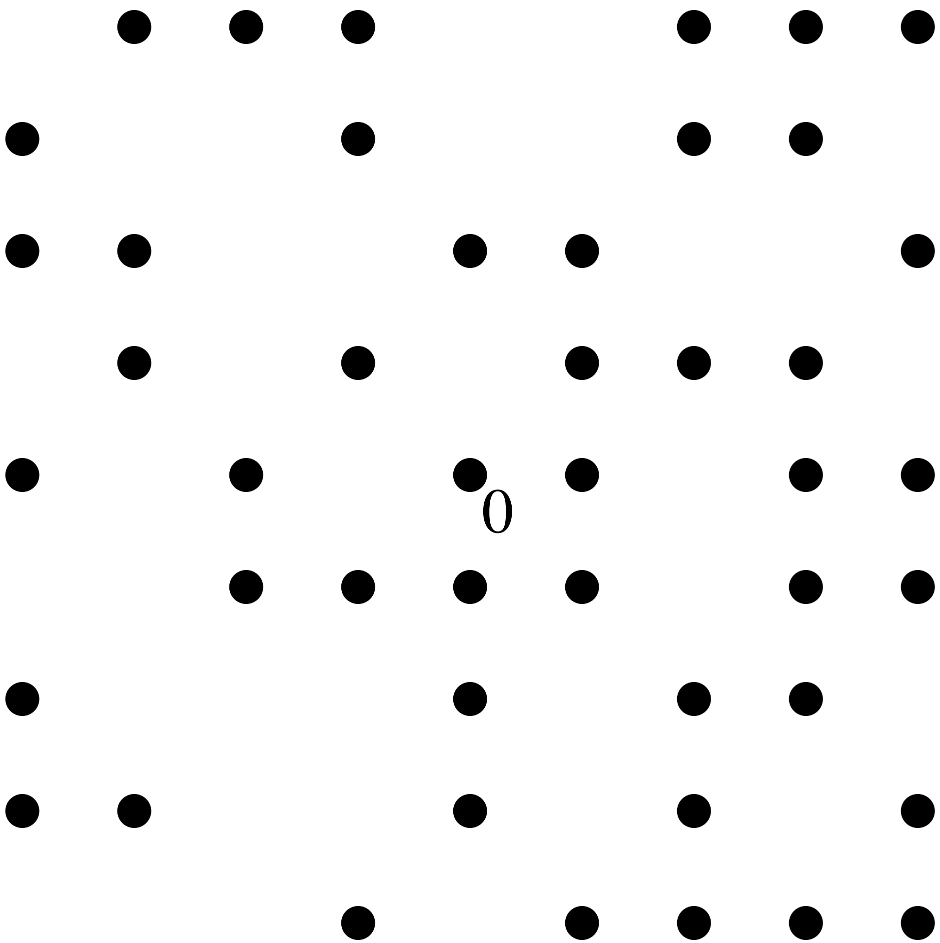


- Infinite lattice
- Each edge is « open » with probability p (i.i.d)
- Let C be the connected component containing 0.
- What is the probability

  $$\theta(p) = P(|C| = \infty) ?$$

- There exists p_c such that
  - $\theta(p) = 0$ if $p < p_c$
  - $\theta(p) > 0$ if $p > p_c$

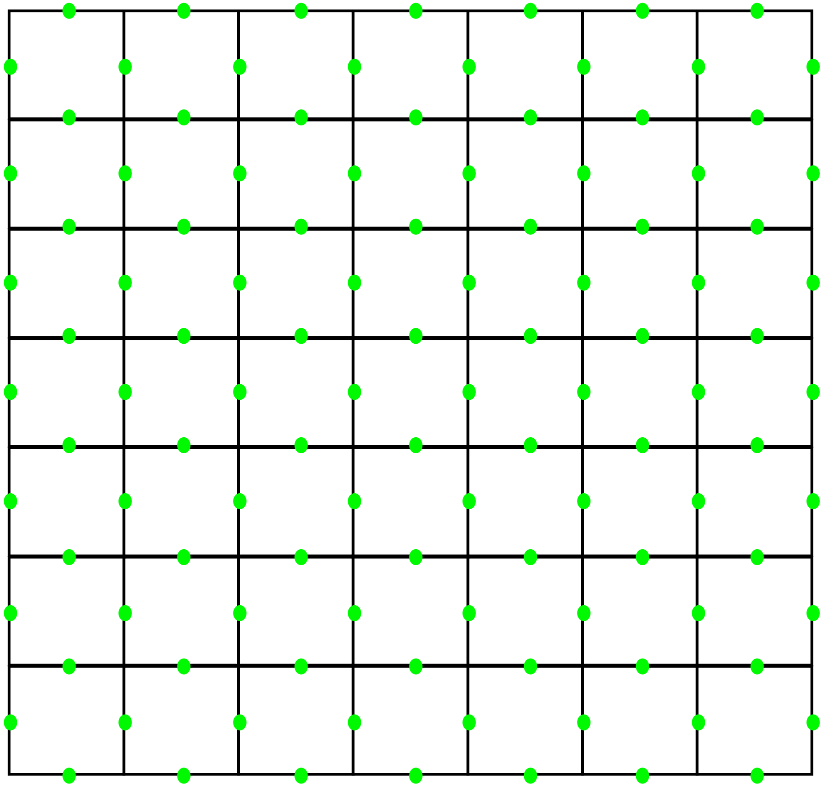
Lattice site percolation

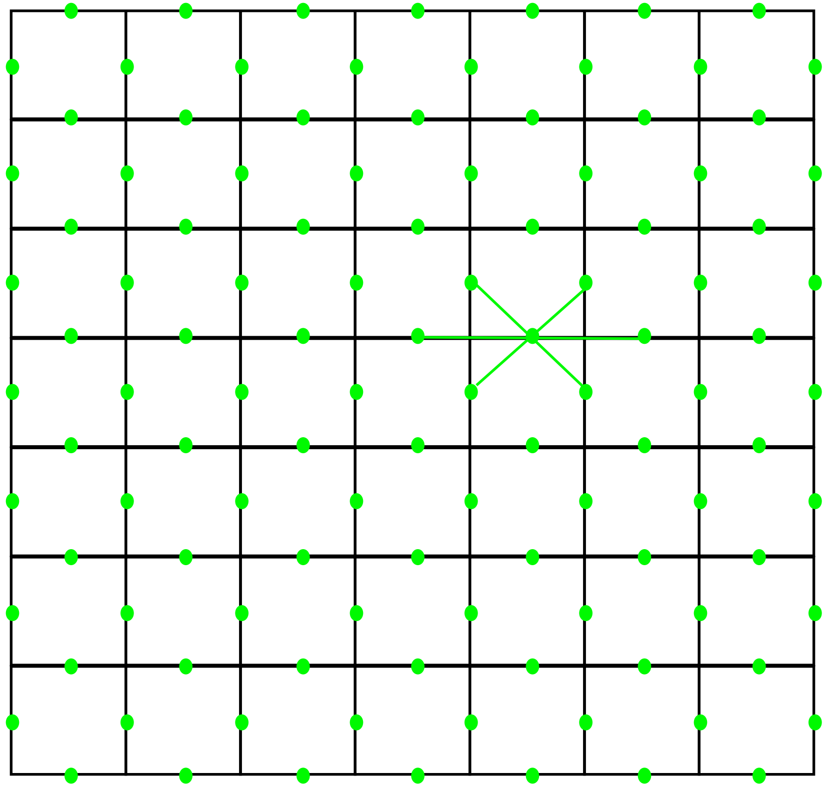


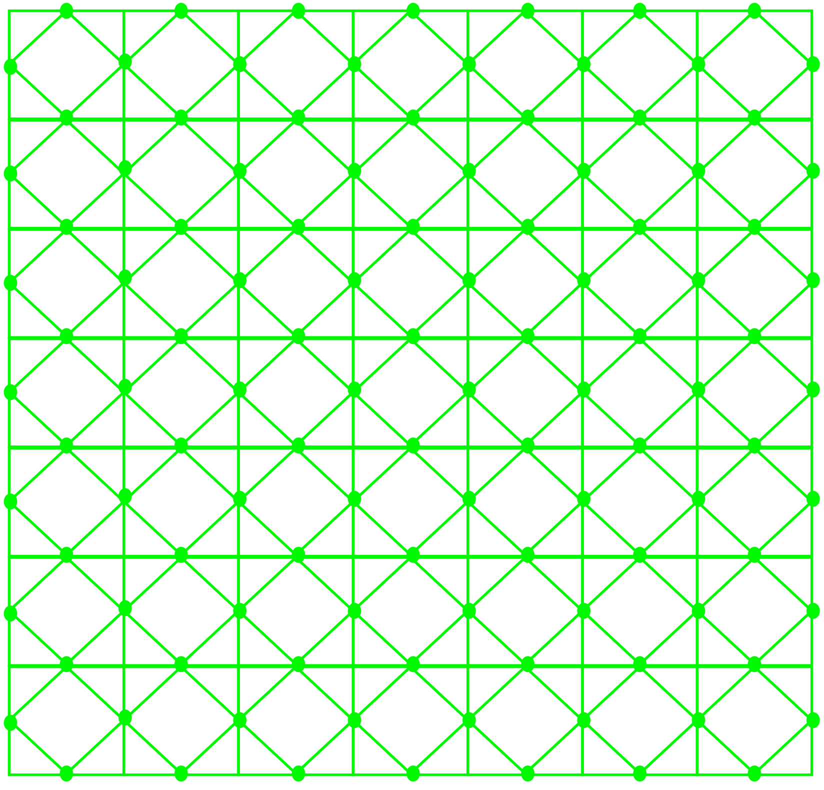
- Infinite lattice
- Each site (vertex) is « occupied » with probability p (i.i.d)
- Let C be the connected component containing 0 (two adjacent occupied sites are connected).
- What is the probability

$$\theta(p) = P(|C| = \infty) ?$$

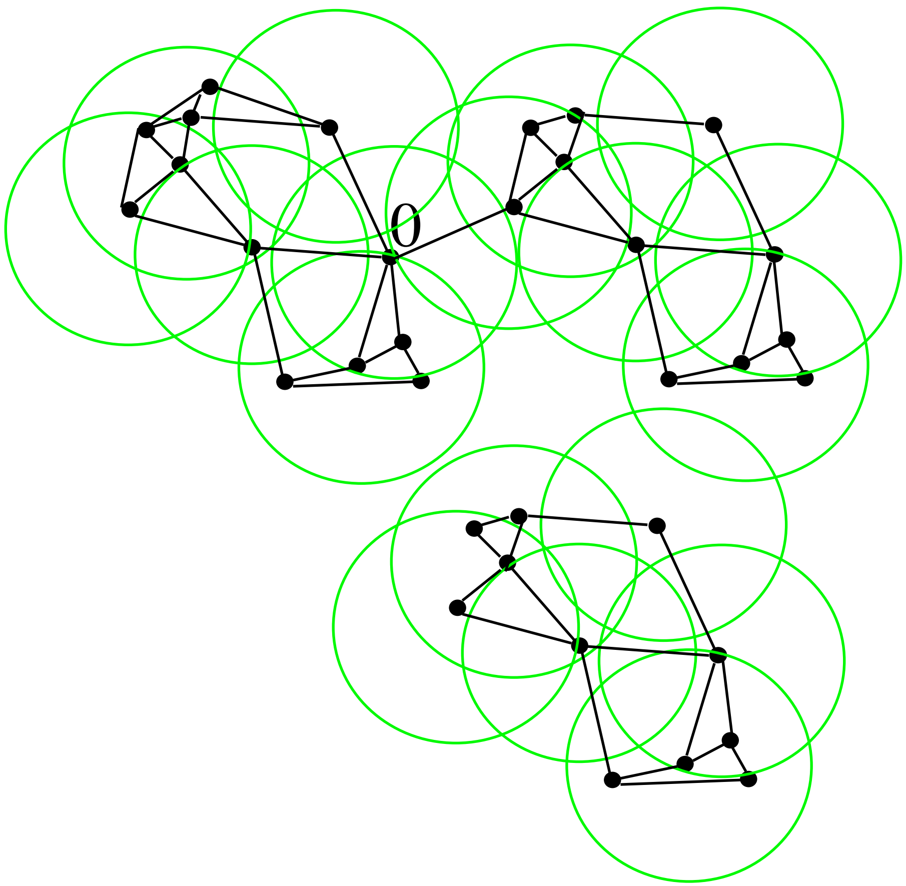
- There exists p_c such that
  - $\theta(p) = 0$ if $p < p_c$
  - $\theta(p) > 0$ if $p > p_c$







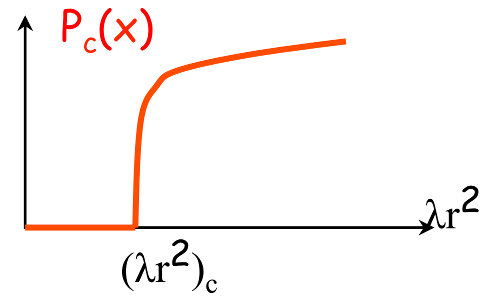
Continuum percolation: Boolean model = RGG



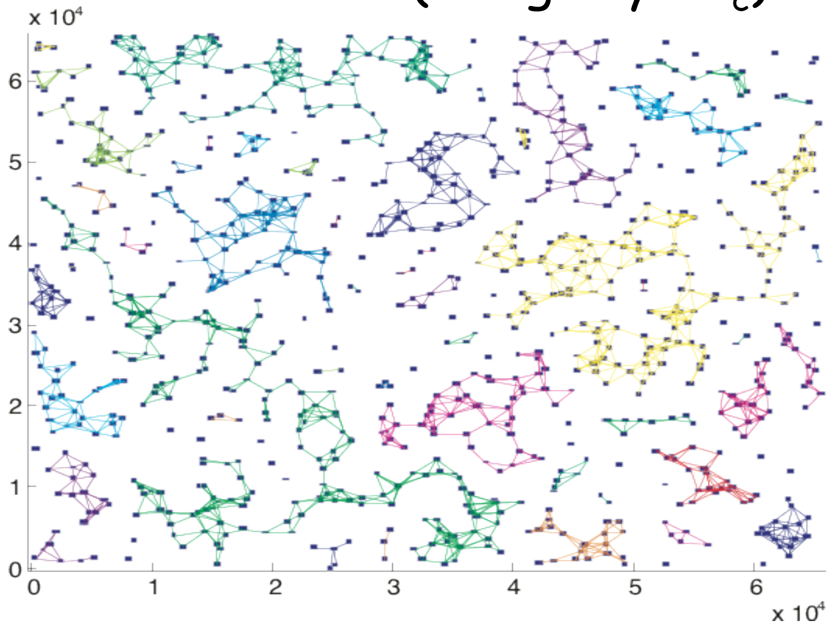
- Position of nodes is a (often homogeneous Poisson) spatial point process, with intensity λ.
- Two nodes are connected to each other if the distance between them is $\leq r$
- This defines a graph $B(\lambda,r)$
- Let C be the connected component containing 0 (two adjacent occupied sites are connected).
- What is the probability $\theta(p) = P(|C| = \infty)$?
- There exists $(\lambda r^2)_c$ such that
  - $\theta(r, \lambda) = 0$ if $r^2\lambda < (r^2\lambda)_c$
  - $\theta(r, \lambda) > 0$ if $r^2\lambda > (r^2\lambda)_c$

Boolean model in the plane

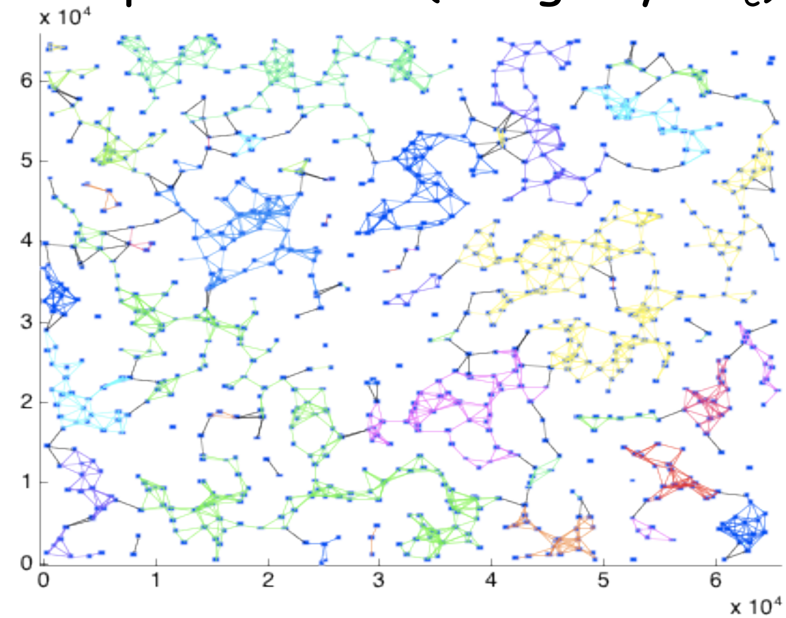
- Percolation theory (see e.g. Meester and Roy 1996): $\Theta(r, \lambda)$ be the probability that an arbitrary node belongs to an infinite cluster (percolation probability). Then there is $(\lambda r^2)_c$ such that
  - $\Theta(r, \lambda) = 0$ if $r^2\lambda < (r^2\lambda)_c$ ("sub-critical")
  - $\Theta(r, \lambda) > 0$ if $r^2\lambda > (r^2\lambda)_c$ ("super-critical")



sub-critical (r slightly < r_c)

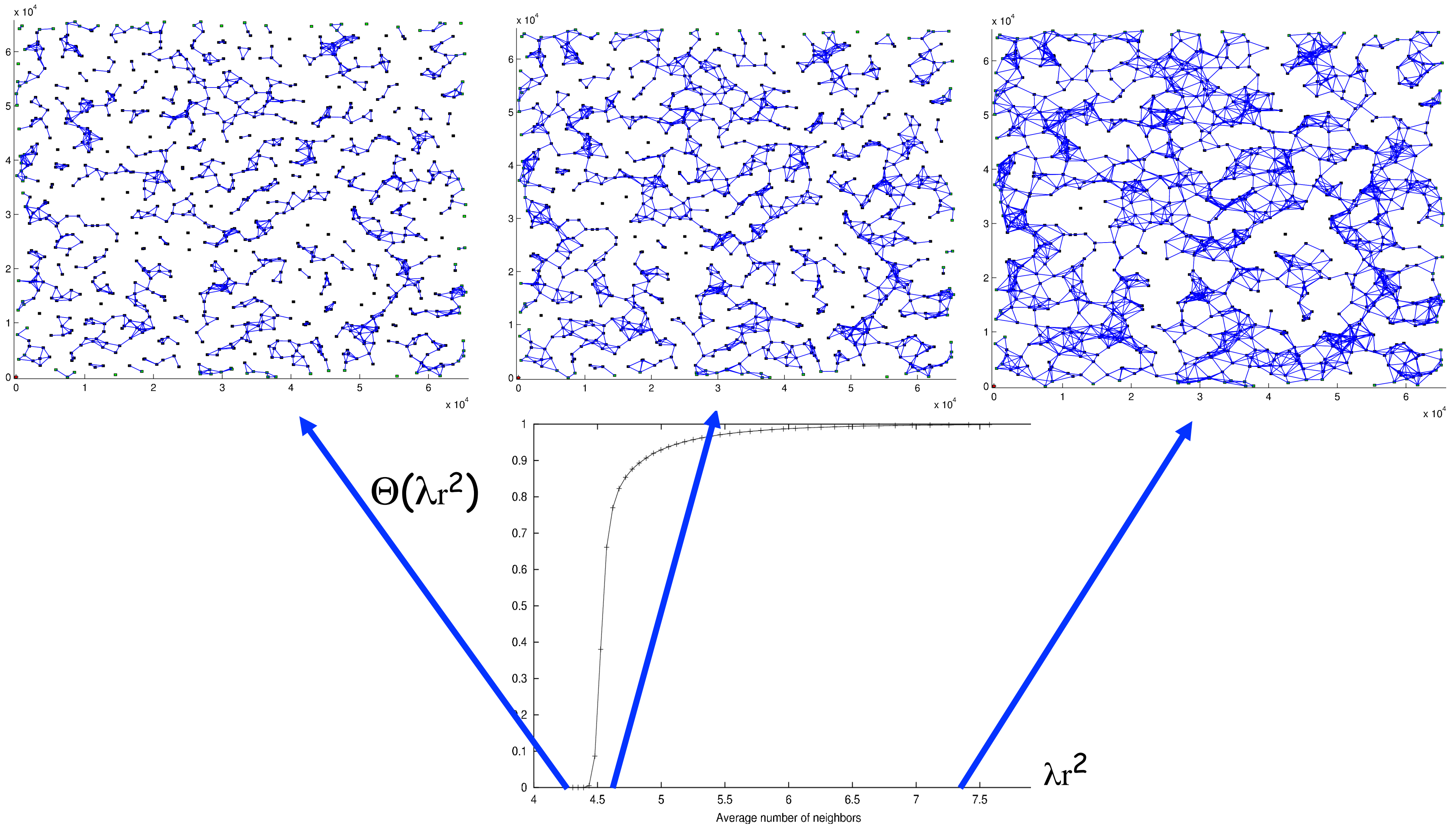


super-critical (r slightly > r_c)



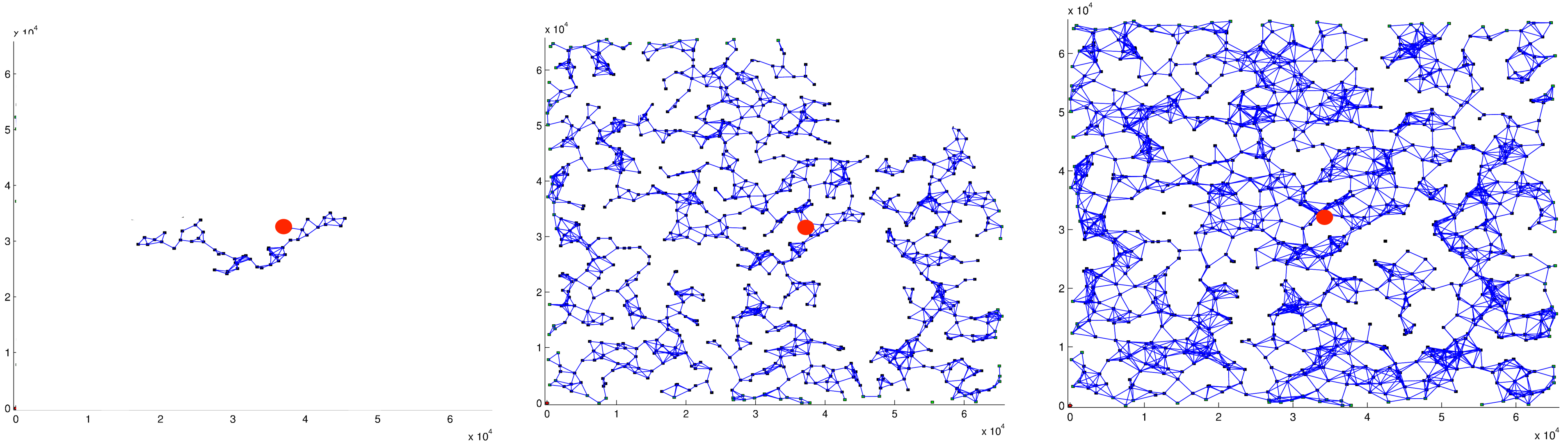
Full or partial connectivity ?

- ❑ Long range connectivity appears much before full connectivity because of a phase transition mechanism (percolation)



Ad hoc or sensor network ?

- Ad hoc network : multiple transmissions, many to many. Connectivity metric = probability that an arbitrary pair of nodes is connected to the rest of the network P_c
- Sensor network : many to one (the base station collecting data). Connectivity metric = probability that one arbitrary node is connected to the base station Θ



- $P_c \approx \Theta^2$ for nodes located far away from each other