Models and Methods for Random Networks: Final Exam

NAME and First Name :

Please write your name on any loose sheets. There are 7 pages. Authorized documents: official class notes only.

Maximum: 50 points

Question 1 (24 points)

Are the following statements true or false? Justify your answer briefly but as accurately, rigorously and completely as possible (if possible, refer to a definition, quote a theorem seen in class, or give a small proof for a true statement; give a comprehensive argument or a counter-example for a false statement).

1. (4 points) Consider the square box $B(n) = [-n, n] \times [-n, n]$ of the 2-dimensional lattice $\mathbb{L}^2$, whose edges are open with probability p independently of each other. Let A_n be the event that there is an open path connecting the left side of $B(n)$ to its right side, and let $\mathbb{P}_p(A_n)$ be the probability that this event occurs. Then for all $0 < p < 1$, $\mathbb{P}_p(A_n)$ decreases with the size of the box n. (T/F)

2. (4 points) Consider bond percolation on the d-dim lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, whose edges are open with probability p independently of each other. Let $\theta(p) = \mathbb{P}_p(|C| = \infty)$, where C is the cluster at the origin (i.e. the part of $\mathbb{L}^d$ containing the set of vertices connected by open paths to the origin and the open edges connecting such vertices). Pick any two vertices $x, y \in \mathbb{Z}^d$. Then for all $0 < p < 1$, the probability that x and y are connected by an open path is at least $\theta^2(p)$. (T/F)

3. (4 points) Let G' and G'' be two random graphs with fixed degree distributions. The two degree distributions have the same mean and different variances. For each graph, let z be the probability that a node reached by an epidemic infects one of its direct neighbors. Remember that the epidemic threshold is the value of z for which the epidemic propagates to a giant component of the graph. If the variance of the node degree of G' is smaller than the variance of the node degree of G'', then the epidemic threshold of G' is also smaller than the epidemic threshold of G''. (T/F)

4. (4 points) Consider two graphs $H_{1,2}$, with $E_1 = \{(1,2),(1,3),(1,4)\}$, and $E_2 = \{(1,2),(2,3),(3,4)\}$. In a $G(n,p)$, there are asymptotically more subgraphs isomorphic to H_1 than to H_2. (T/F)

5. (8 points) We generate two random graphs G_1 and G_2 as follows. For G_1, we generate a $G(n,p)$ with $np = 3$, from which we randomly delete half of the edges. For G_2, we generate a $G(n,p)$ with $np = 3/4$; then we repeat the following step until G_2 has exactly n edges: randomly select a pair of nodes (u,v) that possess a common neighbor w, and add an edge (u,v).

   - Given this construction, G_1 has a giant component (a.a.s.). (T/F)

   - G_2's largest component is larger than G_1's largest component (a.a.s.). (T/F)

Question 2 (9 points)

Consider bond percolation on the *3-dimensional* lattice $\mathbb{L}^3$, whose edges are open with probability p independently of each other. Let p_c be the percolation threshold of $\mathbb{L}^3$.

1. (5 points) Prove that $p_c > 0$. More precisely, find $p_{\min} > 0$, as high as possible, such that you can prove that $p_c \geq p_{\min}$. The higher the value of $p_{\min}$ you find, the higher your score at this question; but the proof must be complete and correct.

2. (4 points) Prove that $p_c < 1$. More precisely, find $p_{\max} < 1$, as low as possible, such that you can prove that $p_c \leq p_{\max}$. The lower the value of $p_{\max}$ you find, the higher your score at this question; but the proof must be complete and correct.

Question 3 (10 points)

1. (4 points) For a random regular graph $G = G(n, 4)$, compute the (asymptotic) probability that G has one 4-cycle and two 5-cycles. Also compute this probability conditionally on G not having any triangles.

2. (2 points) In the same graph G, consider two specific nodes u and v. Compute the number of edges that lie on disjoint paths $u \leftrightarrow v$.

3. (4 points) Suppose we randomly delete k edges from G, and we are interested in the event that all of the disjoint paths from the previous subquestion get interrupted (have at least one edge deleted) as a result. Can you find a threshold function $k(n)$ such that when $k = o(k(n))$, then almost surely u and v remain connected, while when $k = \omega(k(n))$, then almost surely u and v get disconnected?

Question 4 (7 points)

We saw the Kleinberg navigation model in class, where we add random shortcuts to a square lattice. Each shortcut was selected i.i.d. with the same power law of exponent γ. The key result was that efficient local navigation is possible in this graph only with distance exponent $\gamma = 2$.

In this problem, we change the model slightly: for each node u, we generate two shortcuts: the first shortcut selects a node v uniformly among the nodes at distance at most k from u (where k is a constant); the second shortcut is generated as before (power law with exponent γ). Call the resulting graph G.

1. (2 points) Is the performance of local navigation in G better, worse, or equal compared to the original model? Give a short, informal argument.

2. (3 points) Now we want to prove that in G, when $\gamma \neq 2$, efficient navigation is still not possible, despite having two shortcuts. To prove this, we can consider a graph G' in which local navigation is easier than in G, and show that efficient navigation is not possible even in G'.

   The graph G' is obtained as follows: each node u in G has a shortcut to *every* node v at distance at most k (plus the second shortcut generated according to the power law).

   Now show that for the case $\gamma > 2$, efficient navigation in G' is not possible. Hint: you do not need to redo the whole calculation of the original proof, but explain how the proof needs to be modified for G', and give the resulting order of number of steps.

3. (2 points) Answer the same question for $\gamma < 2$.