Lecture # 1 CM 011

Quantum Physics 3

Asumption: you know well quantum physics 1 and 2, from lectures of prof Savona.

Aim of these lectures:

(i) To discuss a transition from quantum to classical phenomena

Motivation. QM is introduced normally in the following way:

- take classical system, and consider its Hamiltonian.
- then, quantize it: replace momentum p and coordinate x by operators $\hat{p}$ and $\hat{x}$ with certain commutational relation,

$$[\hat{p}, \hat{x}] = -i\hbar$$

Say that the state of the system is a vector in Hilbert space $\mathcal{H}$ (in x representation this is a wave-function $\psi(x)$); introduce probabilistic interpretation of the wave function, and formulate certain rules to calculate the observables.

This creates an impression that quantum physics is based in some way on classical physics. From the point of view of fundamental physics this impression is wrong: it is another way around - the Nature is intrincically quantum, whereas classical physics arrises as an approximation to quantum reality. So, we should understand how these rather abstract notions of quantum mechanics (Hilbert space, wave functions, operators, etc) lead to what we normally observe in our life: 3d space instead of Hilbert space, no operators, no probability, etc.

Besides conceptial interest, the study of the transition from quantum to classical world will allow us to introduce a new formalism, known under the name „semiclassical approximation", which will allow us to solve certain problems in quantum physics which cannot be treated in perturbation theory.

Plan of this Chapter:

- From quantum to classial in simple examples
- Formalism of semiclassical approximation

intergange

- Feynman path integral formulation of quantum mechanics

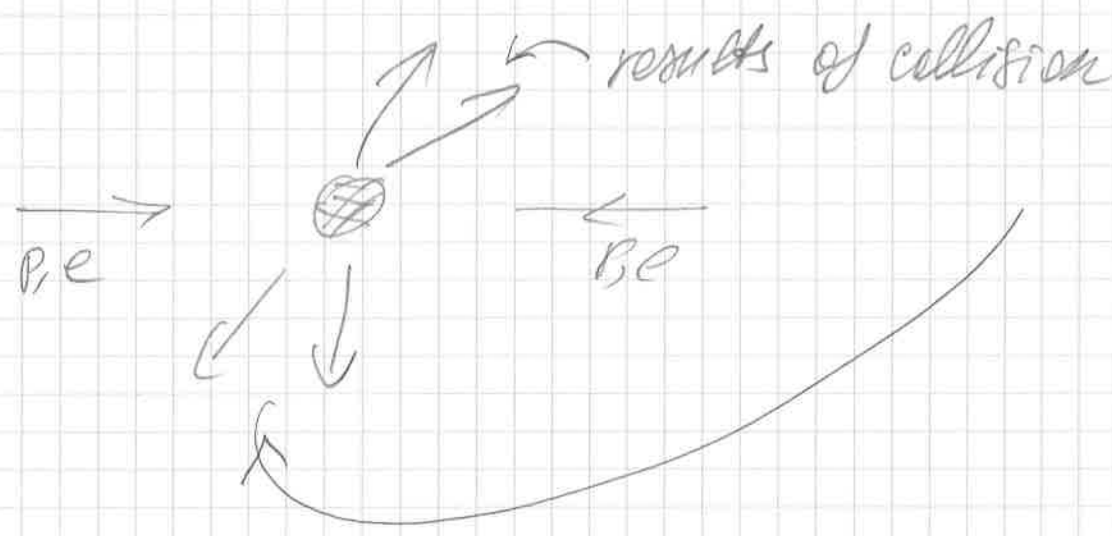
References:
- Mécanique Quantique Cohen-Tannoudji et al
- Landau, Lifshits, Quantum Mechanics, volume 3
- Feynman, Hibbs, Quantum Mechanics and path integrals

- C. Scrucca, Quantum physics 3, EPFL lectures 2012
itp.epfl.ch/page-70378.html

(ii) The second aim of these lectures is to study the theory of scattering in Quantum mechanics

Motivation. Quantum phenomena are essential for small distances and small time scales. How to get to small distances? Collide different particles

Typical scattering experiment:



By studying results of collision we can get information about interactions between (elementary or composite) particles, or even discover new particles. (The most recent discovery is that of the Higgs boson at the LHC).

References: Cohen-Tannoudji; Landau, Scrucca, + J.R. Taylor, Scattering theory: The quantum theory of non-relativistic collisions

(iii) Third aim of these lectures is a short introduction to relativistic quantum mechanics = unification of quantum theory and special relativity. The true unification is in fact Relativistic quantum field theory, see lectures by R. Rattazzi. In my lectures I will follow historical line, so we will discuss Dirac equation, prediction of positron, and relativistic corrections to spectroscopy of a hydroden atom.

References: My lectures in 2012/2013 on QM II, itp.epfl.ch/page-60676.html

Schedule of the lectures & seminars:

Lectures, every Tuesday, $13^{15}-15^{00}$

Seminars, every Friday, $13^{15}-15^{00}$

Assistants: Andrey Shkerin, PhD student at LPPC + Tokareva Anna, postdoc.

2014 Javier Rubio, postdoc at LPPC

2015 Mohamed Anber, ...

page on Moodle:

moodle.epfl.ch/enrol/index.php?id=14069

please put your remarks about lectures and exersices, so that we can adapt/improve our presentations.

Exam: oral, two parts

- theory question
- exersice to solve

] ticket by chance

30 min for preparation, 30 min interrogation. 50% of the grade: theory, 50% - exersice

„Shut up and calculate"

David Mermin

Often attributed to Dirac and Feynman

Chapter 1. Semiclassical approximation

Our starting point:

Quantum theory:

(i) state = vector ψ in Hilbert space

(ii) observable: hermitean operator $\hat{O}$ in Hilbert space

(iii) Result of a measuriment:

$$O = \langle \psi | \hat{O} | \psi \rangle$$

(iv) time evolution: Schrödinger equation,

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H\psi$$

+ all extra usual postulates of Quantum mechanics

How to get from here classical physics:

(i) state: position x and momentum p of a particle

(ii) observable = result of measuriment - some function of p and x

(iii) Time evolution: Newton equation,

$$m\ddot{\vec{x}} = \vec{F}$$

1. From quantum to classical on simple examples.

The simplest systems you have considered in QM I:

- free particle
- harmonic oscillator

Now, we will take these systems from quantum point of view and see how classical description can appear

(i) Free particle (this example you have already studied in QM I, I will go through it just for warm up)

Quantum mechanics:

$$H = \frac{\hat{p}^2}{2m} \; ;$$

eigenvectors, in x representation,

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} ; \quad H\psi_p(x) = \frac{p^2}{2m}\psi_p(x)$$

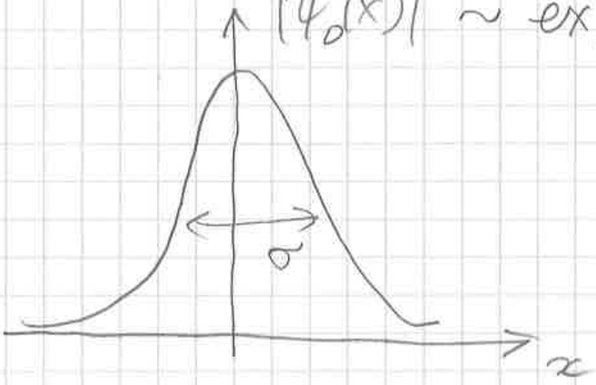
The state $\psi_p(x)$ is very far from what we would call a particle in classical physics

$|\psi_p(x)|^2 = \text{Const}$ – particle is everywhere!

Let us take a normalizable wave function, Gaussian distribution:

$$\psi_0(x) = \frac{1}{\sqrt{\sigma}(2\pi)^{1/4}} e^{ip_0 x/\hbar - \frac{x^2}{4\sigma^2}}$$

$$|\psi_0(x)|^2 \sim \exp\left(-\frac{x^2}{2\sigma^2}\right)$$



Equivalent, in p representation:

$$\psi_0(p) = \frac{1}{\sqrt{2\pi\hbar}}\int dx\, e^{-ipx/\hbar}\psi_0(x) =$$

$$= \sqrt{\frac{\sigma}{\hbar}}\left(\frac{2}{\pi}\right)^{1/4} \exp\left(-\frac{\sigma^2}{\hbar^2}(p-p_0)^2\right)$$

Properties:

$$\langle\hat{x}\rangle = 0 \; ; \quad \langle\hat{p}\rangle = p_0$$

$$\langle\hat{x}^2\rangle = \sigma^2 \; ; \quad \langle(\hat{p}-p_0)^2\rangle = \frac{\hbar^2}{4\sigma^2}$$

This state looks like a particle in classical physics, if the accuracy in measuring coordinate is worse than $\delta x \simeq \sigma$ and accuracy in measuring momentum is worse than $\delta p \simeq \frac{\hbar}{2\sigma}$

Time evolution:

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H\psi \; ; \quad \psi\big|_{t=0} = \psi_0(x)$$

Solution:

$$\psi(x,t) = \frac{\hbar\sqrt{\sigma}}{|\sigma(t)|(2\pi)^{1/4}} \cdot \exp\left(-\frac{i}{2}\arctan\left(\frac{\hbar t}{2m\sigma^2}\right) - \frac{ip_0^2 t}{2m\hbar}\right)$$

$$\cdot\, e^{ip_0x/\hbar - \frac{(x-p_0t/m)^2}{4\sigma^2(t)}}$$

$$\sigma^2(t) = \sigma^2 + \frac{1}{2}\frac{i\hbar t}{m}$$

$$|\sigma^4(t)| = \sigma^4 + \frac{1}{4}\frac{\hbar^2 t^2}{m^2}$$

What happened :

$$\langle \hat{x} \rangle = \frac{p_0 t}{m} \quad ; \quad \langle \hat{p} \rangle = p_0$$

$$\left\langle \left(\hat{x} - \frac{p_0 t}{m} \right)^2 \right\rangle = |\sigma(t)|^4 / \sigma^2 = \sigma^2 + \frac{1}{4} \frac{\hbar^2 t^2}{m^2 \sigma^2}$$

Average values of $\hat{x}$ and $\hat{p}$ are the same as they would be for a classical free particle,

whereas the uncertainty in position grows with time,

$$\delta x \simeq |\sigma(t)| = \sigma \left(1 + \frac{\hbar^2 t^2}{4 \sigma^2 m^2} \right)^{1/2}$$

Conclusions: for free particle we can get classical physics out of quantum for

$|\sigma(t)| \lesssim$ accuracy in x determination!

and $\left| \frac{\hbar}{2\sigma(t)} \right| \lesssim$ accuracy in momentum determination.

(ii) <u>quasi-classical states of the harmonic oscillator.</u>

Quantum description, reminder

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 ; \quad [\hat{x}, \hat{p}] = i\hbar$$

Creation and annihilation operators:

$$a = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}} \hat{x} + i \frac{1}{\sqrt{m\hbar\omega}} \hat{p} \right)$$

$$a^+ = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}} \hat{x} - i \frac{1}{\sqrt{m\hbar\omega}} \hat{p} \right)$$

$$[a, a^+] = 1,$$

$$H = \hbar\omega \left(a^+ a + \frac{1}{2} \right)$$

„vacuum": $a|0\rangle = 0$

eigenstates:

$$|n\rangle = \psi_n = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

Eigen modes:

$$H \psi_n = \hbar\omega \left(n + \frac{1}{2} \right)$$

$$\langle \psi_n | x | \psi_n \rangle = \langle \psi_n | p | \psi_n \rangle = 0$$

Very much different from the classical picture!

Classical picture:

$$\frac{dx}{dt} = \frac{p(t)}{m} \quad ; \quad \frac{dp}{dt} = -m\omega^2 x(t)$$

or, eq for x: $\frac{d^2x}{dt^2} = -\omega^2 x$

Solution :

$$x = x_0 \sin(\omega t + \varphi)$$

x_0 and φ are determined by initial conditions.

Question: what is the quantum state which reproduces the classical picture?

It cannot be the eigenvector of $\mathcal{H}$: - no time dependence!

Let us consider time evolution of creation and annihilation operators in Heisenberg picture of Quantum Mechanics:

$$-\frac{\hbar}{i}\frac{da}{dt} = [a, H] = \hbar\omega a \Rightarrow$$

$$a(t) = a(0)e^{-i\omega t}$$

$$a^+(t) = a^+(0)e^{+i\omega t}$$

Then for $\hat{x}$ and $\hat{p}$ we will get:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a(t) + a^{+}(t)\right) =$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(a(0)e^{-i\omega t} + a^{+}(0)e^{i\omega t}\right)$$

$$\hat{p}(t) = \sqrt{\frac{m\hbar\omega}{2}}\,\frac{1}{i}\left(a(0)e^{-i\omega t} - a^{+}(0)e^{i\omega t}\right)$$

Of course, Hamiltonian is time-independent,

$$H(t) = H(0) = \hbar\omega\left(a^{+}(t)a(t) + \frac{1}{2}\right) =$$

$$= \hbar\omega\left(a(0)^{+}a(0) + \frac{1}{2}\right).$$

Commutational relation between $a(t)$ and $a^{+}(t)$ also remain unchanged,

$$[a(t), a^{+}(t)] = 1.$$

Quantum solutions are almost identical with classical solutions for similar combinations of x and p:

$$\alpha = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x + i\,\frac{1}{\sqrt{m\hbar\omega}}\,p\right)$$

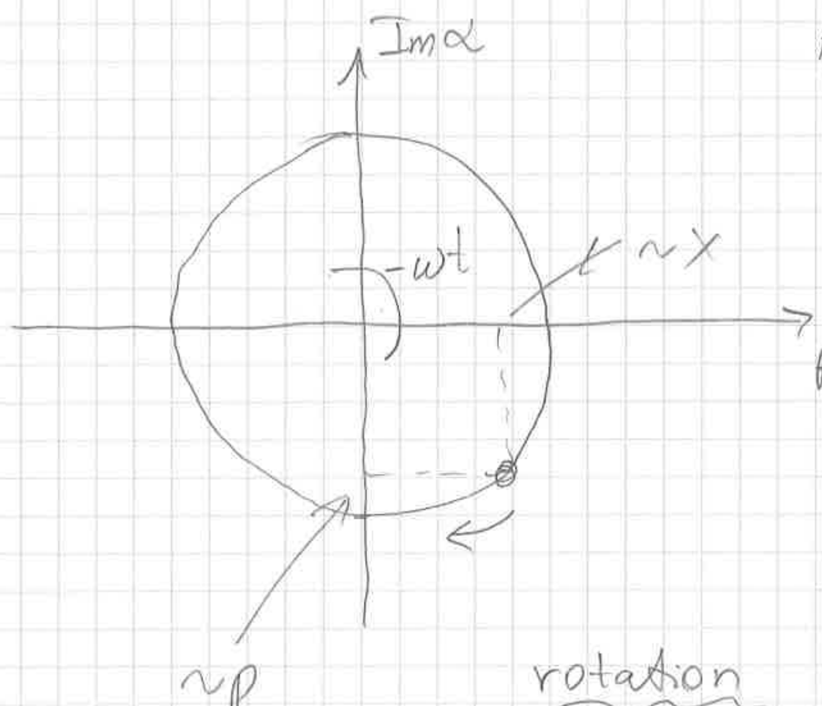
all classical, „c"-number

$$\frac{d\alpha}{dt} = -i\omega\alpha(t); \quad \alpha(t) = \alpha_0 e^{-i\omega t};$$

$$\alpha_0 = \frac{1}{\sqrt{2}}\left(\ldots x(0) + i \ldots p(0)\right)$$

the same coefficients as above.

Visualisation:



Re α ∝ x

Im α ∝ p

Energy in classical physics:

$$H = \frac{1}{2m}\left(p(0)\right)^2 + \frac{1}{2} m\omega^2 \left(x(0)\right)^2 = \hbar\omega |\alpha_0|^2$$

Classical approximation: $H \gg \hbar\omega \Rightarrow$

$|\alpha_0| \gg 1$.

Main observation: dependence of α on time in classical physics is mathematically the same as of the operator a in quantum mechanics.

Let us take some state ψ_0 and consider

$$\langle \hat{x}(t) \rangle \equiv \langle \psi_0 | \hat{x}(t) | \psi_0 \rangle$$

and

$$\langle \hat{p}(t) \rangle \equiv \langle \psi_0 | \hat{p}(t) | \psi_0 \rangle$$

Obviously,

$$\langle \hat{x}(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\langle a(0) \rangle e^{-i\omega t} + \langle a^{+}(0) \rangle e^{i\omega t} \right)$$

$$\langle \hat{p}(t) \rangle = \sqrt{\frac{m\hbar\omega}{2}} \frac{1}{i} \left(\langle a(0) \rangle e^{-i\omega t} - \langle a^{+}(0) \rangle e^{i\omega t} \right)$$

this is exactly what we have in classical physics if we identify

$\langle a(0) \rangle$ with α_0, or $\langle \Psi_0 | a | \Psi_0 \rangle = \alpha_0$

Extra condition: equality of classical energy and quantum energy:

$$\langle \Psi_0 | H | \Psi_0 \rangle = \hbar\omega \langle \Psi_0 | a^+(0) a(0) | \Psi_0 \rangle$$

$$+ \frac{1}{2}\hbar\omega = \hbar\omega |\alpha_0|^2$$

$\Leftarrow$ this term we neglect, as we are in the „classical" regime, $|\alpha_0| \gg 1$.

Together:

$$\begin{cases} \langle \Psi_0 | a | \Psi_0 \rangle = \alpha_0 \\ \langle \Psi_0 | a^+ a | \Psi_0 \rangle = |\alpha_0|^2 \end{cases}.$$

Now, our aim is to determine vector $|\Psi_0\rangle$ for any given α_0.