Neural Networks and Biological Modeling

Professor Wulfram Gerstner
Laboratory of Computational Neuroscience

Question set 8

Exercise 1: Continuous population model

We study a system with lateral connection $w(x-y)$ given by:

$$\tau \frac{\partial h(x,t)}{\partial t} = -h(x,t) + \int w(x-y) F[h(y,t)] dy + I_{ext}(x,t), \tag{1}$$

where $F[h(x,t)] = A(x,t)$ is the population's activity at the point x at time t.

1.1 Show that, for a constant current I_{ext}, there exists a homogeneous stationary solution $h(x,t) = h_0$ with a constant activity A_0 given by:

$$A_0 = F(h_0) = \frac{h_0 - I_{ext}}{\bar{w}},$$

with $\bar{w} = \int w(x-y)dy$.

1.2 We set $h(x,t) = h_0 + \Delta h(x,t)$ where Δh is a small perturbation. Linearize equation (1) around h_0 by substituting $h(x,t) = h_0 + \Delta h(x,t)$ and Taylor-expanding $F[h(x,t)]$ until first order. Apply a spatial Fourier transform (with respect to x) and use the convolution theorem to simplify the resulting temporal differential equation. Solve this differential equation and perform the inverse Fourier transform on the solution to obtain $\Delta h(x,t) = \int g(k) dk$ where

$$g(k) = C(k) e^{ikx} e^{-\kappa(k)t/\tau}.$$

Identify the function $\kappa(k)$. For which values of k do we get $\kappa < 0$? What does this mean for the stability of the solution $\Delta h(x,t)$?

1.3 Consider:

$$w(z) = \frac{\sigma_2 e^{-z^2/(2\sigma_1^2)} - \sigma_1 e^{-z^2/(2\sigma_2^2)}}{\sigma_2 - \sigma_1},$$

with $\sigma_1 = 1$ and $\sigma_2 = 10$. Sketch the qualitative behaviour of $w(z)$ and

$$\int_{-\infty}^{+\infty} w(z) \cos(kz) dz.$$

Determine graphically the stability condition.

hint: $\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2\sigma^2}} cos(kz) dz = \sqrt{2\pi}\sigma e^{-\frac{k^2\sigma^2}{2}}$

Exercise 2: Stationary state in a network with lateral connections

Consider a neural network with lateral connections represented in figure 1: the interaction is locally excitatory and long range inhibitory:

$$w(x,x') = \begin{cases} 1 & |x-x'| \leq \sigma \\ -b & |x-x'| > \sigma \end{cases} \tag{2}$$

Therefore σ corresponds to the range of the excitatory connections. The activity A of a neuron at position x is given by:

$$A(x) = F[h(x)], \tag{3}$$

where $h(x)$ is the total potential of the neuron at position x, defined as:

$$h(x) = \int w(x, x')A(x')dx' + I_{ext}(x). \tag{4}$$

The function $F(h)$ is a simple threshold function:

$$F(h) = \begin{cases} 1 & h > \Theta \\ 0 & h \leq \Theta \end{cases} \tag{5}$$

In this exercise we do not add any external input i.e. $I_{ext}(x) = 0$. The aim of the exercise is to find the neural activity $A(x)$. In order to do so, we assume that $A(x)$ may have a rectangular shape (of breadth $2d$ and amplitude 1, as shown in figure 1) and we prove this assumption with the following passages.

2.1 Consider a point at location x_0 close to $x = 2d$ and calculate its input potential, assuming that $2d > \sigma$.

(Hint: there is excitatory input from the right and there is excitatory and inhibitory input from the left).

2.2 Exploit that at $x_0 = 2d$ we must have $h(x_0) = \Theta$. Why? Calculate d.

2.3 Convince yourselves that the bump of size $2d$ is therefore a solution for the activity $A(x)$ and discuss its properties.

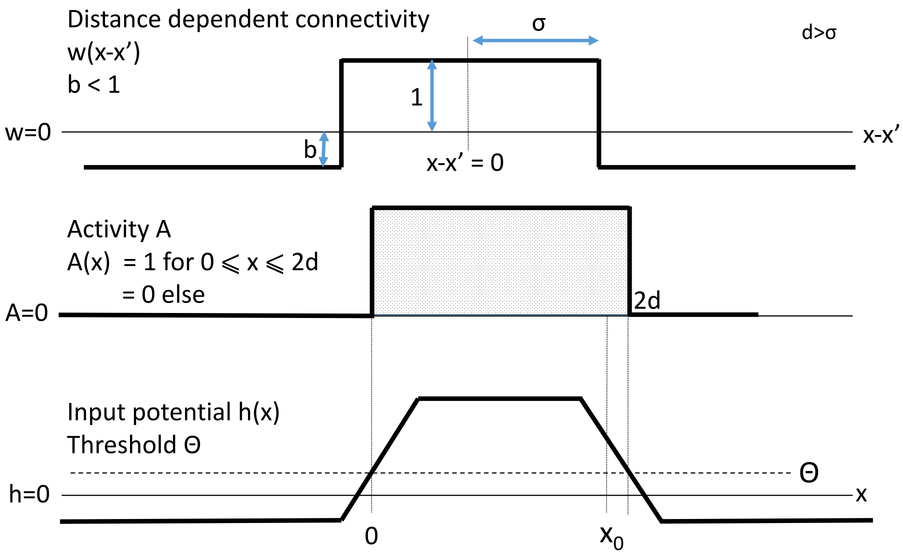

Figure 1: Spatial structure of the network.

Exercise 3: Stability of the stationary bump solution

We keep the same type of network as before and assume that the network rests in the stationary bump solution $A(x)$ determined in the previous exercise. We now investigate the stability of that solution. In order to do so, we perturb the system at time t_p by slightly changing the width $D(t)$ of the activity bump

$A(x, t_p)$ from $2d$ to $2d + \delta$, with some $\delta = \delta(t_p) \ll d$ as indicated in fig. 2. Use the following steps to study the stability of the unperturbed solution:

3.1 Discretize time in the underlying differential equation,

$$\tau \frac{\partial h(x,t)}{\partial t} = -h(x,t) + \int w(x-y) F[h(y,t)] dy, \tag{6}$$

using $\Delta t = \tau$. Solve for $h(x, t + \Delta t)$ on the left hand side.

3.2 Using the discretized equation, calculate the potential $h(x, t_p + \Delta t)$ one time step after the pertubation by explicitly evaluating the integral with the perturbed (broadened) activation $A(x, t_p)$. Consider again a position x_0 close to $x = 2d$

3.3 Evaluate the potential at $x_0 = 2d + \delta$. Is it below or above the threshold Θ? What does this mean for the evolution of the bump width $D(t)$ in the next time step(s)? Based on this, discuss the stability of the stationary solution $A(x)$.

3.4 Derive an iteration formula for the perturbation length $\delta(t_p + \Delta t)$ after one time step. Justify (e.g. via complete induction) that the derived formula holds for all following time steps and derive an explicit formula for $\delta(t_p + n \cdot \Delta t)$, where n is the number of time steps. What is the asymptotic value of δ for $n \to \infty$ and what is the functional form of the underlying decay?

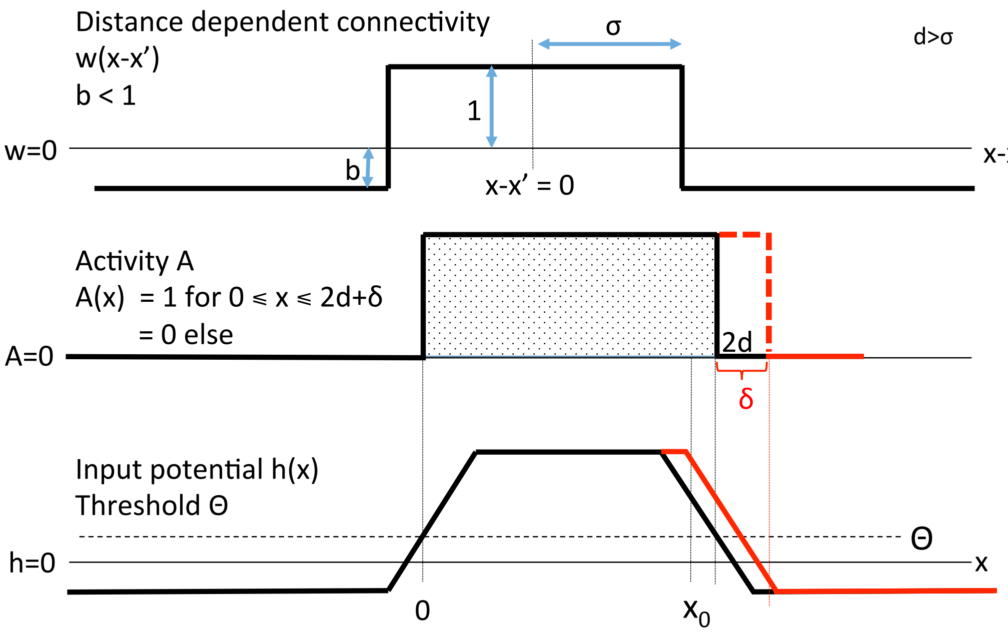

Figure 2: Illustration of the slight pertubation of the activity bump.