23.09.2014

Lecture # 2 , Quantum physics 3

Main points of Lecture # 1

- True, fundamental physics is intrinsically quantum and deals with vectors in Hilbert space, operators, and Schrödinger equation. In some limit, it should lead to classical physics, based on c-number variables p and x, and deterministic evolution described by Newton equation

- This is indeed the case for free particle: quantum state describing „classical" evolution is not a proper state of H, but is a Gaussian packet, which moves according to classical physics laws, but gets larger

- For harmonic oscillator we found that :

(i) for any state ψ the average values of coordinate $\hat{x}$ and momentum $\hat{p}$ follow classical equations of motion. The correspondence is given by

$$\langle \psi | a(0) | \psi \rangle = \alpha(0)$$

where $\alpha(0)$ is related to classical initial conditions as

$$\alpha(0) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\, x(0) + i \frac{1}{\sqrt{m\omega\hbar}}\, p(0) \right),$$

and $a(0)$ is quantum creation operator.

(ii) If the state ψ satisfies also a condition

$\langle \psi | a^{+} a | \psi \rangle = |\alpha(0)|^2$, then the classical energy $H = \hbar\omega |\alpha|^2$ is equal to quantum energy

$$\langle \psi | H | \psi \rangle = \hbar\omega\left(\langle \psi | a^{+} a | \psi \rangle + \frac{1}{2} \right)$$

up to a small correction $\frac{1}{2}\hbar\omega$

(2a)

This is summarized here

Quantum	Classical
$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$	The same without ^
$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right)$	$a \to \alpha$; $\hat{x} \Rightarrow x$; $\hat{p} \to p$
$H = \hbar\omega(a^+a + \frac{1}{2})$	$H = \hbar\omega\alpha^*\alpha$
$[a, a^+] = 1$	$[\alpha, \alpha^*] = 0$
Time dependence, Heisenberg picture $a(t) = a(0)e^{-i\omega t}$	Time dependence via eq. of motion $a \to \alpha$
$\langle p \rangle \equiv \langle \psi \vert \hat{p} \vert \psi \rangle$; $\langle x \rangle = \langle \psi \vert \hat{x} \vert \psi \rangle$	
$\frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}$; $\frac{d\langle p \rangle}{dt} = -m\omega^2\langle x \rangle$	The same

Definition of coherent state, closest to classical state:

$\langle \psi \vert a \vert \psi \rangle = \alpha$ (time evolution of average values in QM is the same as in classical physics)

$\langle \psi \vert a^+a \vert \psi \rangle = \vert\alpha\vert^2$ (the same energy up to $\frac{1}{2}\hbar\omega$)

Plan for today:

(i) Determine ψ from equations

$$\langle\psi|a|\psi\rangle=\alpha_0$$

$$\langle\psi|a^{+}a|\psi\rangle=|\alpha|^2$$

(ii) Study the properties of „coherent" state ψ

(i) To find ψ, let us consider operators

$$b=a-\alpha_0$$

$$b^{+}=a^{+}-\alpha_0^{*}$$

Take a square b^+b:

$$b^+b = a^+a - \alpha_0 a^+ - \alpha_0^* a + \alpha_0^* \alpha_0$$

and take matrix element

$$\langle \Psi_0 | b^+ b | \Psi_0 \rangle = \underset{\langle a^+a \rangle}{|\alpha_0|^2} - \underset{-\alpha_0 \langle a^+ \rangle}{\alpha_0 \alpha_0^*} - \underset{\alpha_0^* \langle a \rangle}{\alpha_0^* \alpha_0} + \underset{\langle\,\rangle}{\alpha_0^* \alpha_0} = 0$$

Since $\langle \Psi_0 | b^+ b | \Psi_0 \rangle$ is the norm of the vector $b|\Psi_0\rangle$, then $b|\Psi_0\rangle = 0 \Rightarrow$

$$(a - \alpha_0)|\Psi_0\rangle = 0 \Rightarrow a|\Psi_0\rangle = \alpha_0 |\Psi_0\rangle.$$

So, $|\Psi_0\rangle$ is eigenvector of annihilation operator with eigenvalue α_0!

To find it explicitely, decompose $|\Psi_0\rangle \equiv |\alpha\rangle$ with the use of eigenvectors of H:

$$|\alpha\rangle = \sum_n C_n(\alpha) \cdot |n\rangle = \sum_n C_n \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} C_n \frac{1}{\sqrt{n!}} a \cdot (a^+)^n |0\rangle =$$

$$\left[a (a^+)^n |0\rangle = n (a^+)^{n-1} |0\rangle \right] \leftarrow \text{note that}$$

$$= \sum_{n=1}^{\infty} c_n \frac{1}{\sqrt{n!}} n(a^+)^{n-1} \cdot |0\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} \, |n-1\rangle =$$

~~equation~~ $$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} \, |n\rangle$$

$\hat{a}|\alpha\rangle = \alpha_0 |\alpha\rangle$ gives:

$$c_{n+1}\sqrt{n+1} = \alpha_0 c_n \quad , \text{ or}$$

$$c_{n+1} = \frac{\alpha_0}{\sqrt{n+1}} c_n \Rightarrow$$

$$c_1 = \frac{\alpha_0}{1} c_0 \; ; \quad \text{etc} \;, \text{ leading to}$$

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \Rightarrow |\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

$$= c_0 \exp[+\alpha a^+] \, |0\rangle$$

c_0 can be found from normalisation condition, $\langle\alpha|\alpha\rangle = 1$, we fix the phase of c_0 to be zero. Finally,

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha a^+\right)|0\rangle$$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Normalisation of α-state:

we should have

$$|C_0|^2 \langle 0| e^{+\alpha^* a} e^{\alpha a^+} |0\rangle = 0$$

From Glauber f-la on page 9, we have:

$$e^{\alpha^* a} e^{\alpha a^+} = e^{\alpha^* a + \alpha a^+} e^{\frac{1}{2}\alpha^* \alpha}$$

$$e^{\alpha a^+} e^{\alpha^* a} = e^{\alpha^* a + \alpha a^+} e^{-\frac{1}{2}\alpha^* \alpha}$$

So,

$$e^{\alpha^* a} e^{\alpha a^+} = e^{\alpha a^+} e^{\alpha^* a} \exp[|\alpha|^2] \Rightarrow$$

$$|C_0|^2 \langle 0| e^{\alpha^* a} e^{\alpha a^+} |0\rangle = |C_0|^2 \exp[|\alpha|^2] \cdot$$

$$\cdot \langle 0| e^{\alpha a^+} e^{\alpha^* a} |0\rangle = |C_0|^2 \exp|\alpha|^2 \Rightarrow$$

$$C_0 = \exp[-\tfrac{1}{2}|\alpha|^2]$$

phase is not fixed, we take it real.

Properties of α - states:

Trivial ; - $\langle\alpha|H|\alpha\rangle = \hbar\omega\left[|\alpha|^2 + \frac{1}{2}\right]$

- behaviour of $\langle\hat{p}\rangle$ and of $\langle\alpha|\hat{x}|\alpha\rangle$ is the same as in classical physics

Less trivial :

Probability to have n - state :

$$\mathcal{P}_n = e^{-|\alpha|^2} \cdot \frac{|\alpha|^{2n}}{n!} \underset{\text{large } n}{\approx} \frac{1}{\sqrt{2\pi n}} \exp\left[-n\log n + n + 2n\log|\alpha| - |\alpha|^2\right]$$

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

maximum : $\frac{\partial}{\partial n}\left[-n\log n + n + 2n\log|\alpha| - |\alpha|^2\right] =$

$$= -\log n - \cancel{1} + \cancel{1} + 2\log|\alpha| = 0 \Rightarrow n = |\alpha|^2 \Rightarrow$$

$$\mathcal{P}_n \approx \frac{1}{\sqrt{2\pi|\alpha|^2}} \exp\left[\cancel{-|\alpha|^2\log|\alpha|^2} + \cancel{|\alpha|^2} + \cancel{2|\alpha|^2\log\alpha} - \cancel{|\alpha|^2} - \frac{1}{2}\frac{(n-|\alpha|^2)^2}{|\alpha|^2}\right] \approx$$

dispersion:

$$\frac{\partial^2}{\partial n^2}\left[\quad\right] = -\frac{1}{n}$$

$$\approx \frac{1}{\sqrt{2\pi|\alpha|^2}} \exp\left[-\frac{1}{2}\frac{(n-|\alpha|^2)^2}{|\alpha|^2}\right]$$

Gaussian distribution with a peak in $n = |\alpha|^2$ (as expected) and with a width

$$\delta n \sim |\alpha| \ll |\alpha|^2 \text{ for large } |\alpha| :$$

Energy is well determined!

Uncertainty in coordinate and momentum:

$$\langle\alpha|x|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\,\alpha = \sqrt{\frac{2\hbar}{m\omega}}\, \frac{\alpha+\alpha^*}{2}$$

$$\langle\alpha|p|\alpha\rangle = \sqrt{2m\hbar\omega}\, \mathrm{Im}\,\alpha = \sqrt{2m\hbar\omega}\, \frac{\alpha-\alpha^*}{2i}$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[4(\mathrm{Re}\,\alpha)^2 + 1\right] ; \quad \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}$$

$$\langle p^2\rangle = \frac{m\hbar\omega}{2}\left[4(\mathrm{Im}\,\alpha)^2 + 1\right] ; \quad \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2}$$

So, $\delta x = \sqrt{\frac{\hbar}{2m\omega}}$; $\delta p = \sqrt{\frac{m\hbar\omega}{2}}$;

$\delta x\, \delta p = \frac{\hbar}{2}$ — does not depend on α and on time!

Wave-function of α-state in x-representation; $\psi_\alpha(x) = \langle x|\alpha\rangle$

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha a^+\right)|0\rangle$$

Let us introduce operator $D(\alpha) = e^{\alpha a^+ - \overset{*}{\alpha} a}$

$$D^+(\alpha) = e^{\overset{*}{\alpha} a - \alpha a^+}$$

This is a unitary operator,

$$D^+ D = D D^+ = 1.$$

Glauber f-la:

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$$

It is valid (only) if $[A,[A,B]] = [B,[A,B]] = 0$

with the use of it, we can write $D(\alpha)$ as:

$$D(\alpha) = e^{\alpha a^+ - \overset{*}{\alpha} a} = e^{\alpha a^+} e^{-\overset{*}{\alpha} a} e^{+\frac{1}{2}[\alpha a^+, \overset{*}{\alpha} a]}$$

$$= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^+} e^{-\overset{*}{\alpha} a}$$

Obviously,

$$e^{-\overset{*}{\alpha} a}|0\rangle = |0\rangle$$

Therefore,

$$D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2 + \alpha a^+}|0\rangle \equiv |\alpha\rangle$$

$D(\alpha)$ converts vacuum to coherent state $|\alpha\rangle$.

Now, we can find the wave function of coherent state in x-representation.

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \langle x|D(\alpha)|0\rangle$$

Let us find $D(\alpha)$ in terms of $\hat{x}$ and $\hat{p}$:

$$D(\alpha) = \exp\left[\sqrt{\frac{m\omega}{\hbar}}\left(\frac{\alpha - \alpha^*}{\sqrt{2}}\right)\hat{x} - \frac{i}{\sqrt{m\hbar\omega}}\left(\frac{\alpha + \alpha^*}{\sqrt{2}}\right)\hat{p}\right]$$

$=$ (use Glauber f-la) $=$

$$\exp\left[\sqrt{\frac{m\omega}{\hbar}}\,\frac{\alpha - \alpha^*}{\sqrt{2}}\,\hat{x}\right]\exp\left[-\frac{i}{\sqrt{m\hbar\omega}}\,\frac{\alpha + \alpha^*}{\sqrt{2}}\,\hat{p}\right]$$

$$\cdot \exp\left[\frac{\alpha^{*2} - \alpha^2}{4}\right]$$

So,

$$\langle x|D(\alpha)|0\rangle = \overbrace{\exp\left(\sqrt{\frac{m\omega}{\hbar}}\,\frac{\alpha-\alpha^*}{\sqrt{2}}\,x\right)\cdot\exp\left(\frac{\alpha^{*2}-\alpha^2}{4}\right)}^{A}\cdot$$

$$\cdot\,\langle x|\underbrace{\exp\left(-\frac{i}{\sqrt{m\hbar\omega}}\,\frac{\alpha+\alpha^*}{\sqrt{2}}\,\hat{p}\right)}_{\text{operator of translation}}\cdot|0\rangle =$$

$$= \left\langle x-\sqrt{\frac{\hbar}{2m\omega}}\,(\alpha+\alpha^*)\,\middle|\,0\right\rangle\cdot A$$

for definition of $\langle p\rangle_\alpha$ and $\langle x\rangle_\alpha$ see page 8.

$$= e^{i\theta_\alpha}\,e^{i\langle p\rangle_\alpha x/\hbar}\;\psi_0\left(x-\langle x\rangle_\alpha\right)$$

$$\theta_\alpha = \frac{\alpha^{*2}-\alpha^2}{4i}$$

ψ_0: ground state wave-function of oscillator in x-representation,

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{x^2}{4\delta x^2}\right);$$

$$\delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

Wave function of α- state in x- representation:

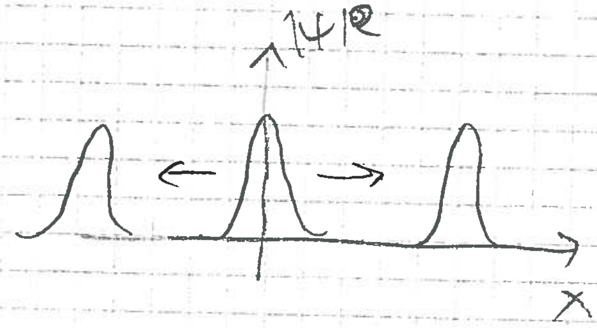
$$\langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\left[\frac{x-\langle x\rangle}{2\delta x}\right]^2 + i\langle p\rangle\frac{x}{\hbar}\right\} e^{i\theta_\alpha}$$

Time dependence: change $\langle p\rangle$ by $\langle p(t)\rangle$ and $\langle x\rangle$ by $\langle x(t)\rangle$, and multiply by $e^{-i\omega t/2}$:

$$\psi(x,t) = \langle x|\psi(t)\rangle =$$

$$= \langle x|\alpha\rangle\Big|_{\substack{\langle x\rangle\to\langle x(t)\rangle \\ \langle p\rangle\to\langle p(t)\rangle}} \cdot e^{-i\omega t/2}$$

wave packet moves without distortion:



$$|\alpha(t)\rangle = e^{-\frac{iHt}{\hbar}} \sum e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum \frac{1}{\sqrt{n!}}|n\rangle\alpha^n \cdot e^{-i\omega(n+1/2)} \Rightarrow e^{-i\omega/2\, t}\, |e^{-i\omega t}\alpha\rangle$$