QUANTUM PHYSICS III

Appendix A: the Fresnel integral formula

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We know the value of the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}}, \quad a > 0 . \tag{1}$$

Here we prove that this result can be extended to the case of imaginary a, that is

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{i\alpha x^2}{2}} = \sqrt{\frac{2\pi}{i\alpha}}, \quad \alpha \in \mathbb{R} . \tag{2}$$

The proof is based on Cauchy's theorem : the integral of an analytic function over a closed contour in the complex plane is zero.

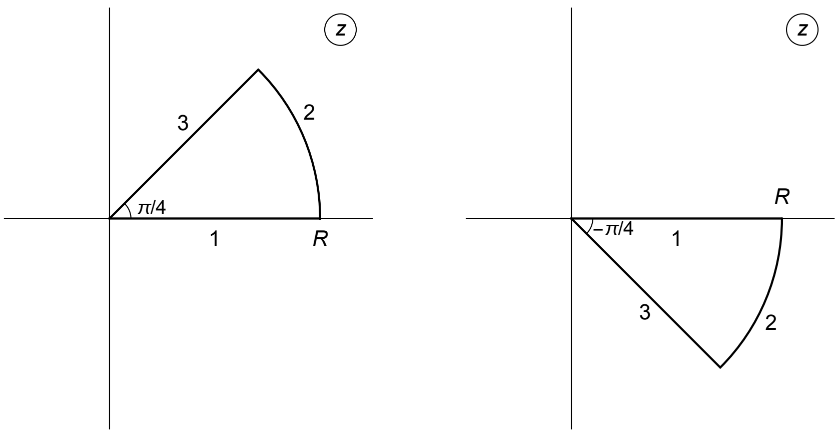

FIGURE 1 – The contours

Consider the integral

$$I = \oint dz\, e^{-\frac{az^2}{2}}, \quad a > 0 , \tag{3}$$

taken over one of the contours shown on figure 1. For example, we take the left contour first. Then,

$$I = I_1 + I_2 + I_3 , \tag{4}$$

where I_i is the part of I computed over the i'th part of the contour. In the limit of large R, the first term becomes the usual Gaussian integral,

$$I_1 = \int_0^R dx\, e^{-\frac{ax^2}{2}} . \tag{5}$$

In the term I_2, one integrates over the arc of radius R. We change the variables as follows,

$$z = Re^{i\phi} , \quad dz = iRe^{i\phi} d\phi . \tag{6}$$

Hence,

$$I_2 = iR \int_0^{\pi/4} d\phi \, e^{-\frac{a}{2}R^2(cos 2\phi + i \sin 2\phi) + i\phi} \, . \tag{7}$$

Finally, in I_3 the change of variables

$$z = y e^{\frac{i\pi}{4}} \tag{8}$$

yields

$$I_3 = -e^{\frac{i\pi}{4}} \int_0^R dy \, e^{-\frac{iay^2}{2}} \, . \tag{9}$$

By Cauchy's theorem,

$$I_1 + I_2 + I_3 = 0 \, . \tag{10}$$

Now, we want to show that in the limit $R \to \infty$, the integral I_2 vanishes. To this end, note first that

$$|I_2| \leqslant \int_0^{\pi/4} d\phi \, \left| e^{-\frac{a}{2}R^2(cos 2\phi + i \sin 2\phi) + i\phi} \right| \leqslant R \int_0^{\pi/4} d\phi \, e^{-\frac{a}{2}R^2 \cos 2\phi} \, . \tag{11}$$

The integrand in the r.h.s. of this inequality vanishes exponentially fast at large R unless ϕ is close to $\pi/4$. To check the last region carefully, we split the integral

$$R \int_0^{\pi/4} d\phi \, e^{-\frac{a}{2}R^2 \cos 2\phi} = R \int_0^{\beta} + R \int_\beta^{\pi/4} , \quad 0 < \beta < \frac{\pi}{4} \, , \tag{12}$$

and make an estimation for the second term,

$$R \int_\beta^{\pi/4} d\phi \, e^{-\frac{a}{2}R^2 \cos 2\phi} < R \int_\beta^{\pi/4} d\phi \, \frac{\sin 2\phi}{\sin 2\beta} e^{-\frac{a}{2}R^2 \cos 2\phi} = \frac{1}{aR \sin 2\beta} \left[e^{-\frac{a}{2}R^2 \cos 2\phi} \right]_{\phi=\beta}^{\phi=\pi/4} \sim \frac{1}{R} \, . \tag{13}$$

Thus, I_2 behaves as $1/R$. From eq. (4) it then follows that

$$\lim_{R \to \infty} (I_1 + I_3) = 0 \, . \tag{14}$$

Hence, from eqs. (1) and (9) we have

$$\int_0^\infty dy \, e^{-\frac{iay^2}{2}} = e^{-\frac{i\pi}{4}} \sqrt{\frac{\pi}{2a}} , \quad a > 0 \, , \tag{15}$$

or

$$\int_{-\infty}^\infty dy \, e^{-\frac{iay^2}{2}} = \sqrt{\frac{2\pi}{ia}} , \quad a > 0 \, . \tag{16}$$

This proves eq. (2) for $\alpha > 0$. To prove it for $\alpha < 0$, it suffices to repeat the steps above for the right contour on figure 1.