Lecture #4,

Addendum: saddle point approximation

1. Real case, 1 dimensional integral

$$\int dx \exp\left[-\frac{1}{\lambda} A(x)\right] \cdot f(x)$$

Problem: find expansion of A when $\lambda \to 0$

Solution: let $A(x) = A(x_0) + \frac{1}{2} A''(x_0)(x-x_0)^2$

$+ \dots$, i.e. x_0 is minimum of $A(x)$:

$A''(x_0) \geq 0$

Then we have

$$\int dx \exp\left[-\frac{1}{\lambda} A(x_0)\right] \exp\left[-\frac{1}{2\lambda} A''(x_0)(x-x_0)^2 + \dots\right]$$

$$\left[f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \dots\right] = *$$

To understand better the expansion parameter, make a change of variables:

$$\frac{A''(x-x_0)^2}{\lambda} = y^2; \quad (x-x_0) = \sqrt{\frac{\lambda}{A''}} \cdot y;$$

$$* = \exp\left[-\frac{1}{\lambda} A(x_0)\right] f(x_0) \cdot \sqrt{\frac{\lambda}{A''}} \int dy \cdot \exp\left(-\frac{1}{2} y^2 - \frac{1}{\lambda} \frac{A'''}{3!} \left(\frac{\lambda}{A''}\right)^{3/2} y^3 - \frac{1}{\lambda} \frac{A''''}{4!} \left(\frac{\lambda}{A''}\right)^2 y^4 + \dots\right)$$

$$\left(1 + \frac{f'}{f}\left(\frac{\lambda}{A''}\right)^{1/2} y + \frac{1}{2!}\frac{f''}{f} \cdot \frac{\lambda}{A''} y^2 + \ldots\right) = *$$

We can now expand exponential in powers of $\sqrt{\lambda}$, and account for the fact that the terms odd in y do not contribute:

$$* = \exp\left[-\frac{1}{\lambda} A(x_0)\right] f(x_0) \sqrt{\frac{\lambda}{A''}} \cdot \int_{-\infty}^{+\infty} dy \, \exp\left(-\frac{1}{2} y^2\right) \cdot$$

$$\cdot \left[1 + \frac{1}{2!}\frac{f''}{fA''} \cdot \lambda \cdot y^2 - \frac{1}{4!}\frac{A''''}{(A'')^2} \lambda \cdot y^4 \right. \qquad (*)$$

$$+ \frac{1}{2!}\frac{(A''')^2}{(A'')^3}\left(\frac{1}{3!}\right)^2 \lambda \cdot y^6 - \frac{A'''}{3!}\frac{1}{(A'')^2}\frac{f'}{f} \lambda \cdot y^4 +$$

$$\left. O(\lambda^2)\right] \approx \sqrt{\frac{2\pi\lambda}{A''}} f(x_0) e^{-\frac{A(x_0)}{\lambda}} \left(1 + O(\lambda)\right)$$

2. Pure imaginary case

$$\int dx \, \exp\left[\frac{i}{\lambda} A(x)\right] f(x), \qquad \lambda \to +0$$

with $A(x)$ – real.

Steps: regulate the integral as

$$\int dx \, \exp\left[\frac{i}{\lambda} A(x) - \varepsilon |x|\right] f(x),$$ and

expand as previously in the real case.

Question: why should we expand at point x_0 when $A'(x_0)=0$?

Let us try to expand in arbitrary x_0:

$$\int dx \left[f(x_0)+f'(x_0)(x-x_0)+\ldots\right]\cdot$$
$$\exp\left[\frac{i}{\lambda}A(x_0)+\frac{i}{\lambda}A'(x_0)(x-x_0)+\frac{i}{\lambda}\frac{1}{2!}A''(x_0)(x-x_0)^2+\ldots\right.$$

Leading term, Gaussian part:

$$f(x_0)\exp\left(\frac{i}{\lambda}A(x_0)\right)\cdot\int dx\,\exp\left[\frac{i}{\lambda}A'(x_0)(x-x_0)+\frac{i}{\lambda}\frac{1}{2!}A''(x_0)\cdot\right.$$
$$\left.\cdot(x-x_0)^2\right]=f(x_0)\exp\left[\frac{i}{\lambda}A(x_0)\right]\cdot$$
$$\cdot(\text{prefactor})\cdot\exp\left[-\frac{i^2}{\lambda^2}\frac{(A')^2}{2\cdot\frac{i}{\lambda 2!}\cdot A''}\right]$$
$$=f(x_0)\exp\left(\frac{i}{\lambda}A(x_0)\right)\cdot(\text{prefactor})\cdot\exp\left[-\frac{i}{\lambda}\frac{(A')^2}{A''}\right]$$

Corrections, from higher orders in $(x-x_0)$, are not suppressed by any power of $\lambda \Rightarrow$ No helpful expansion can be produced if $A'(x_0)\neq 0$.

Question: when we can trust saddle point approximation?

We should require that corrections $O(\lambda)$ are small, i.e. (from(*))

$$\frac{f''}{fA''}\cdot\lambda \ll 1$$

$$\frac{A''''}{(A'')^2}\,\lambda \ll 1$$

$$\frac{(A''')^2}{(A'')^3}\,\lambda \ll 1 \qquad (**)$$

$$\frac{A'''f'}{(A'')^2 f}\cdot\lambda \ll 1$$

Generalisation to the multi-dimensional integrals is straightforward, and the analogues of relations (**) should be satisfied for classical computations to be valid.