Lecture # 7, Quantum physics III

Main points of # 6

- determined matching conditions at the turning points and found applicability of this result

- analysed bound state problem in WKB and found Bohr-Sommerfeld quantisation condition for energy levels,

$$\oint p\,dx = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

Today

- Planck formula, $\Delta E = \hbar\omega$

- Tunneling probability through a potential barrier

- Life-time of a metastable state

Planck f-la: difference between nearby energy levels, following from

$$\oint p\,dx = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

Let us find variation of

$\int_b^a p\,dx$ with respect of energy:

$$p = \sqrt{2m(E-U)}$$

$$\frac{\partial}{\partial E}\int_b^a p\,dx = \underbrace{p(a)\frac{\partial a}{\partial E} - p(b)\frac{\partial b}{\partial E}}_{\text{zero at turning points}} + \int_b^a \frac{\partial p}{\partial E}dx =$$

$$= \int_b^a \sqrt{2m}\,\frac{1}{\sqrt{E-U}}\cdot dx \cdot \frac{1}{2} = \int_b^a \sqrt{\frac{m}{2(E-U)}}\,dx$$

what is the physical meaning of this integral?

$$m\frac{dx}{dt} = \sqrt{2m(E-U)} \Rightarrow m\frac{dx}{\sqrt{2m(E-U)}} = dt \Rightarrow$$

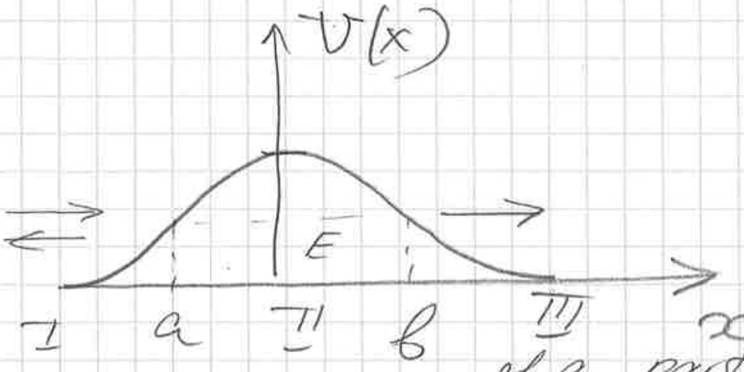
$$\int_b^a \sqrt{\frac{m}{2(E-U)}}\,dx = \frac{T}{2}:$$ half-period of classical oscillations

$$\text{so, } \frac{\partial}{\partial E}\oint p\,dx = T \Rightarrow$$

$$T\Delta E = 2\pi\hbar\,\Delta n \Rightarrow$$

$$\Delta E = \frac{2\pi}{T}\hbar\,\Delta n = \omega\hbar\,\Delta n,$$ Planck formular!

Tunneling probability through a barrier



$V(x) \to 0$ for $x \to \pm\infty$

Let Energy of a particle be smaller than the height of the barrier, and a particle is coming from the left. In classical physics, the particle will come to point a, and then will go back. In quantum physics, it will tunnel through the barrier with some probability. Our aim is to find it.

Wave function for $x \to +\infty$:

$$\psi \to D\exp\left[\frac{i}{\hbar} p x\right] \qquad E = \frac{p^2}{2m};\ p > 0$$

transmitted wave

Wave function for $x \to -\infty$:

$$\psi \to \left\{1 \cdot \exp\left[\frac{i}{\hbar} p x\right] + R\exp\left[-\frac{i}{\hbar} p x\right]\right\}$$

initial wave — reflected wave

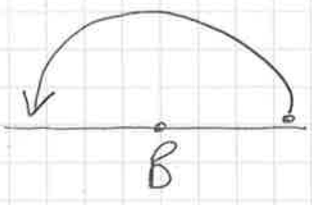
D : transmission amplitude,

R : reflection amplitude, $|R|^2+|D|^2=1$.

In semiclassical approximation, in the region III we can write:

$$\psi = \cdot \frac{1}{\sqrt{p}} \exp\left[\frac{i}{\hbar}\int_b^x p(x')dx'\right]\cdot C$$

Let us make x complex, and



get the wave-function for $x<b$ with the use of analytic continuation:

$$p(x) \approx \sqrt{2m(-U'(b))(x-b)} \; ;$$

$$\int_b^x p(x')dx' = \sqrt{2m(-U'(b))}\cdot\frac{2}{3}(x-b)^{3/2}$$

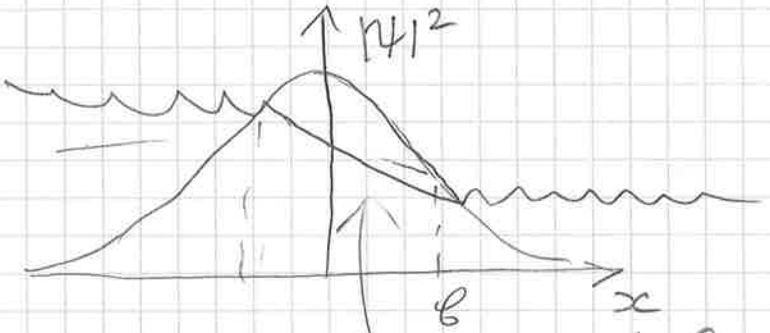
$x-b=\rho e^{i\varphi}$, and φ changes from 0 to π $\Longrightarrow$ $(x-b)^{3/2} \to \rho^{3/2}e^{3i\pi/2} = \rho^{3/2}(-i)$

for $x<b$ we have

$$\psi = e^{-i\pi/4}\frac{1}{\sqrt{|p|}}\exp\left[\frac{1}{\hbar}\int_x^b |p|\,dx'\right]C$$

$$= \underbrace{\frac{1}{\sqrt{|p|}}\,e^{-i\pi/4}\cdot\exp\left[\frac{1}{\hbar}\int_a^b |p|dx'\right.}\left.-\frac{1}{\hbar}\int_a^x |p|\,dx'\right]\cdot C$$

If we would continue analytically via lower complex half-plane, we would get an exponentially small solution which should be discounted in our approximation.



We already know what happens with this wave function: for $x < a$ we must have:

$$\frac{2}{\sqrt{|p|}} \exp\left[-\frac{i\pi}{4}\right] \exp\left[\frac{1}{\hbar}\int_a^b |p|\, dx'\right] \cdot$$

$$\cdot \cos\left(\frac{1}{\hbar}\int_x^a |p|\, dx - \frac{\pi}{4}\right) \cdot C$$

$$= \frac{C}{\sqrt{|p|}} \cdot \exp\left[\frac{1}{\hbar}\int_a^b |p| dx'\right] e^{-i\pi/4}$$

$$\cdot \Big[\underbrace{e^{\frac{i}{\hbar}\int_x^a |p| dx - i\frac{\pi}{4}}}_{\text{initial wave.}} + \underbrace{e^{-\frac{i}{\hbar}\int_x^a |p| dx + \frac{i\pi}{4}}}_{\text{reflected wave}} \Big]$$

Summary:

Scattering asymptotics:

$$x \to +\infty, \quad \psi = D\exp\left(\frac{i}{\hbar} px\right)$$

$$x \to -\infty, \quad \psi = \exp\left[\frac{i}{\hbar} px\right] + R\exp\left[-\frac{i}{\hbar} px\right]$$

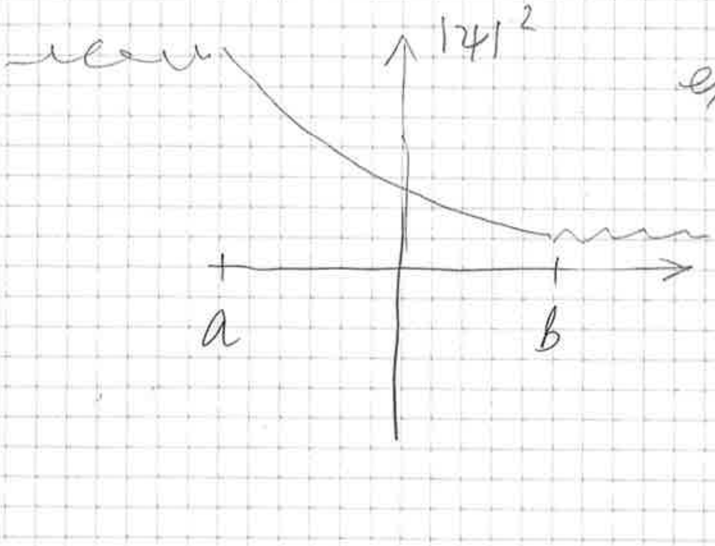
$|D|^2$ transmission probability.

Found:

$$\text{III}: \quad \psi = \frac{C}{\sqrt{p}} \exp\left(\frac{i}{\hbar}\int_b^x p\,dx\right)$$

$$\text{II}: \quad \psi = \frac{C}{\sqrt{|p|}} e^{-i\pi/4} \exp\left[\frac{1}{\hbar}\int_a^b |p|dx\right] \cdot \exp\left[-\frac{1}{\hbar}\int_a^x |p|dx\right]$$

term $\sim \exp\left[-\frac{1}{\hbar}\int_x^b |p|dx\right]$ must be neglected, as it corresponds to exponentially small function at $x \approx a$



exponential growth, if considered from point b, exponential decay, if from point a

Region I :

$$\frac{c}{\sqrt{p}} e^{-i\pi/4} \exp\left[\frac{1}{\hbar}\int_a^b |p| dx\right] \cdot 2\cos\left(\frac{1}{\hbar}\int_x^a p\, dx - \frac{\pi}{4}\right)$$

$$= \frac{c}{\sqrt{p}} \exp\left[\frac{1}{\hbar}\int_a^b |p| dx\right] e^{-i\pi/4} \times$$

$$\left[\underbrace{\exp\left(\frac{i}{\hbar}\int_x^a p\, dx - \frac{i\pi}{4}\right)}_{\text{falling wave}} + \underbrace{\exp\left(-\frac{i}{\hbar}\int_x^a p\, dx + \frac{i\pi}{4}\right)}_{\text{reflected wave}}\right]$$

So, we get that

$$C = \exp\left[-\frac{1}{\hbar}\int_a^b |p|\,dx\right] \cdot e^{i\pi/2}$$

leading to

$$R = \exp\left[i\,\frac{\pi}{2}\right]$$

$$D = e^{i\pi/2} \exp\left[-\frac{1}{\hbar}\int_a^b |p|\,dx\right]$$

Tunneling probability:

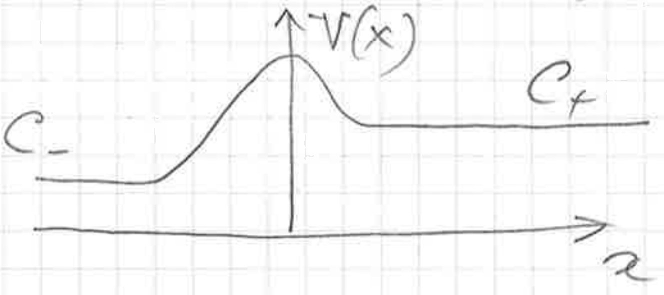
$$|D|^2 = \exp\left[-\frac{2}{\hbar}\int_a^b |p|\,dx\right]$$

in this approximation $|R|^2 = 1$ — exponentially small terms cannot be traced.

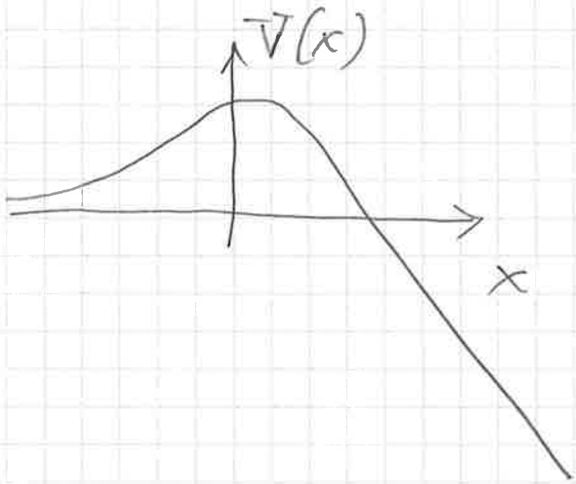
As usual, the conditions of applicability of the semiclassical approximation and of the analysis of the turning point must be met.

Remark: in the problem we considered in the lecture the asymptotics of the potential were the same as $x \to \pm\infty$. In fact, the situation may be more general. For example,

$$V(x \to \pm\infty) = C_\pm, \quad C_+ \neq C_-$$



Also, potential at $\pm\infty$ is not necessary a constant, e.g.



Still, the tunneling probability can be defined.

Life-time of a metastable state

Important examples:

Radioactive α-decay

$[1\,MeV]^{-1} = 6{,}58 \cdot 10^{-22}\,s$

$${}^{238}_{92}U \rightarrow {}^{234}_{90}Th + \alpha$$

uranium thorium helium, ${}^{4}_{2}He$

$$M_U > M_{Th} + M_\alpha \; ; \; (M_U - M_{Th} - M_\alpha) \simeq 5\,MeV$$

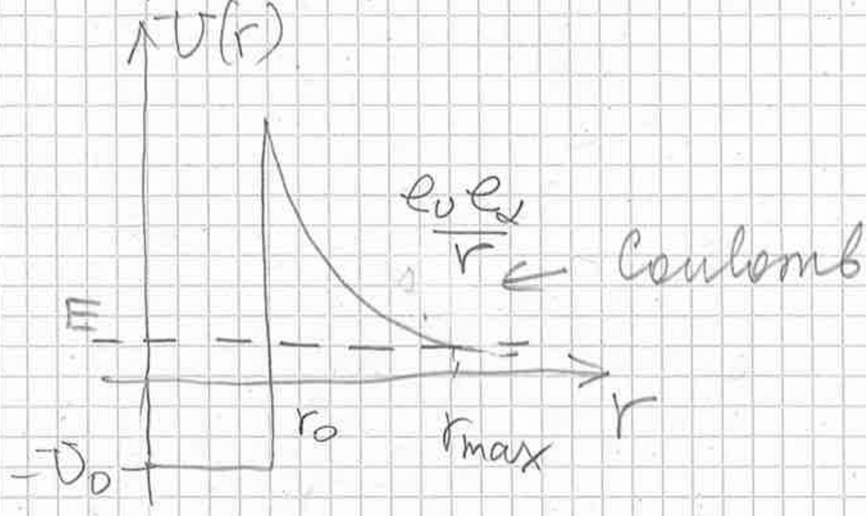
$$\tau_0 = 1.4 \cdot 10^{17}\,s \simeq 4.5 \cdot 10^{9}\ \text{years}$$

How do understand this long life-time?
(Gamow, 1928; Gurney & Condon, 1929)

Nuclear forces: attraction of α and Th at small distances.

Electomagnetic forces: repulsion of Th and α at large distances.

Potential, as a function of distance, of Th and α-particles, very schematically



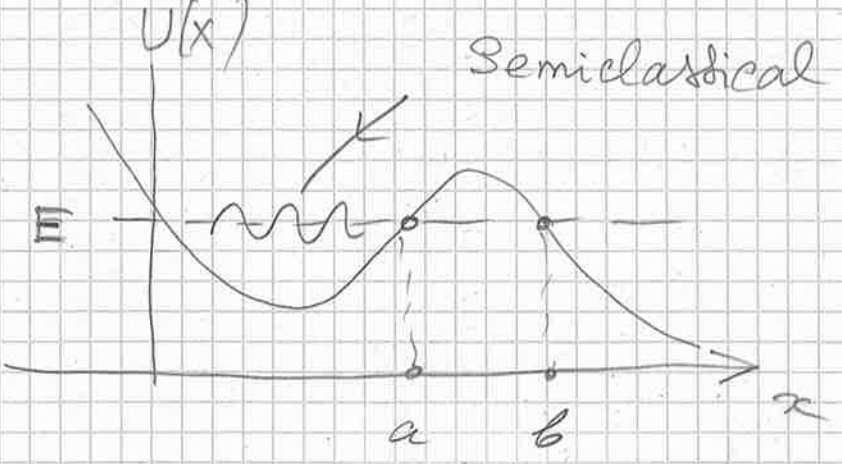
r_0: size of the nucleus

U_0: binding energy of α-particle

E: total energy of the α-particle inside nucleus.

How to determine life-time of the "metastable" state?

1D computation:



Probability to tunnel through the barrier, transmission coefficient:

$$|D|^2 = \exp\left[-\frac{2}{\hbar}\int_a^b |p|\,dx\right]$$

Probability to tunnel in unit time:

$$\frac{dP}{dt} = \frac{1}{T}\cdot|D|^2 ,$$

, where T is classical period of osillations of particle in the classically alowed region between a and b.

This leads to the life-time

$$\tau = T|D|^{-2} = T\exp\left[+\frac{2}{\hbar}\int_a^b |p|\,dx\right].$$

Application to Uranium decay:

$$|D|^2 \simeq \exp\left[-\frac{2}{\hbar}\int_{r_0}^{\frac{e_U e_\alpha}{E}} \sqrt{2m_\alpha\left(\frac{e_U e_\alpha}{r} - E\right)}\,dr\right] =$$

turning point: $E = \frac{e_U e_\alpha}{r_{max}} \Rightarrow r_{max} = \frac{e_U e_\alpha}{E}$

$$= \exp\left\{-\frac{2\beta}{\hbar}\sqrt{\frac{2m_\alpha}{E}}\left(\arccos\sqrt{\frac{Er_0}{\beta}} - \sqrt{\frac{Er_0}{\beta}\left(1-\frac{Er_0}{\beta}\right)}\right)\right\}$$

where $\beta = e_T e_\alpha$

To make a very dirty and quick estimate let us take $r_0 \to 0$.

Then

$$|D|^2 = \exp\left[-\frac{\pi\beta}{\hbar}\sqrt{\frac{2m}{E}}\right] = \exp\left(-\frac{2\pi\beta}{\hbar v}\right),$$

where v is the velocity of α-particle after decay.

$\beta = e_T \cdot e_\alpha = \hbar c \alpha \cdot N_T N_\alpha$

α - fine structure constant, N_T & N_α are the number of protons in Th and α, $N_T = 90$, $\alpha = 2$.

Then, numerically,

$$|D|^2 = \exp\left(-\frac{2\pi\alpha \cdot 90 \cdot 2 \cdot c}{v}\right) = \exp\left(-8.25\frac{c}{v}\right)$$

$\frac{mv^2}{2} = 5$ MeV $\Rightarrow$ $v = 0.05$, leading to $|D|^{-2} \simeq \exp(165) \simeq 10^{71}$:

too large number is compare with

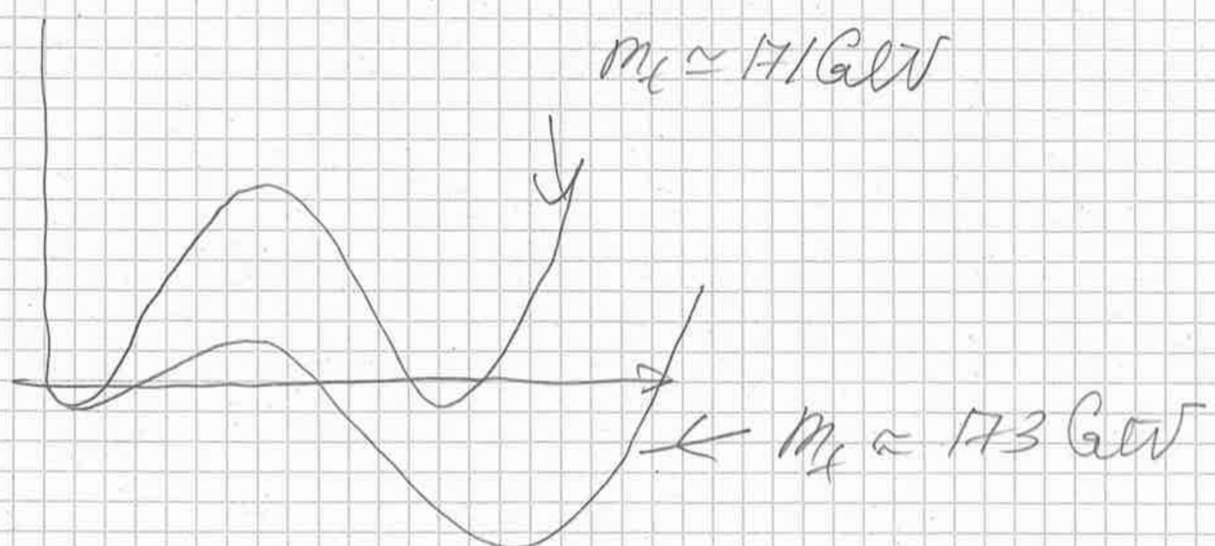
experiment. Indeed, to get the dimensions right, we should have

$$\tau \sim \frac{r_0}{v} |D|^{-2} \simeq 10^{-22} s \cdot 10^{71} \simeq 10^{49} s$$

but experiment is $\simeq 10^{17}$ s. Our estimate was too rough – factor 3 off in the exponential.

<u>Another</u> interesting example of a metastable state: our vacuum.

The potential for the Higgs scalar field (think about is as a coordinate x) has the following form, depending on the „mass" of the top quark:



Our vacuum may be absolutely stable, or may be metastable, depending on the mass of the top quark, which is still not known with sufficient accuracy. In the worst case scenario

$$\tau \simeq 10^{80} \tau_{universe}.$$