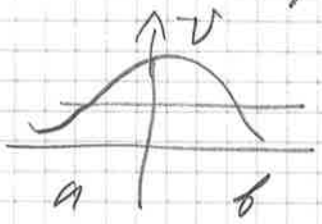
Lecture #8, Quantum physics 3

Main points of

- Found probability of tunneling.



$$\mathcal{P} = \exp\left[-\frac{2}{\hbar}\int_a^b |p|\,dx\right]$$

- discussed decay of a metastable state,

$$\tau = T\,\mathcal{P}^{-1}$$

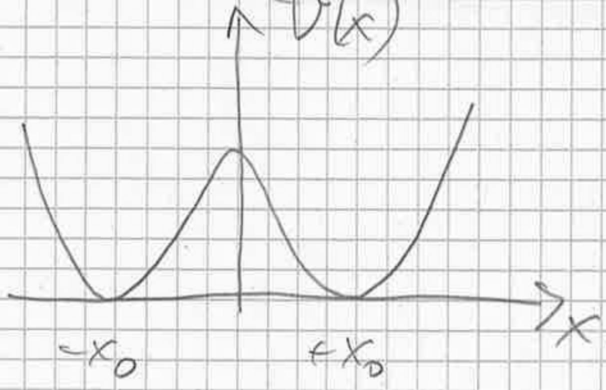
Plan

- Semiclassical energy splitting
- New topic - scattering theory

Semiclassical energy-level splitting

Consider double-well potential, e.g.

$$V(x) = \lambda\left(x^2 - x_0^2\right)^2.$$



Many applications

- Maser (Ammonia) (Nobel prize, Townes)

Particle physics - instantons, spontaneous symmetry breaking.

Classically: the states with equal energy, oscillating near $x = x_0$; and another near $x = -x_0$.

Quantum-mechanically:

theorem – it is impossible to have degenerate states in 1D quantum systems. Proof: suppose this is not the case, i.e. we have ψ_1 & ψ_2:

$$H\psi_1 = E\psi_1 \; ; \quad H\psi_2 = E\psi_2 \; ; \quad \psi_1/\psi_2 \neq \text{Const}$$

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \Rightarrow$$

$$\left.\begin{aligned} \psi_1'' &= \frac{2m}{\hbar^2}(V-E)\psi_1 \\ \psi_2'' &= \frac{2m}{\hbar^2}(V-E)\psi_2 \end{aligned}\right\} \Rightarrow \frac{\psi_1''}{\psi_1} = \frac{\psi_2''}{\psi_2}, \text{ or}$$

$$\psi_1''\psi_2 - \psi_1\psi_2'' = 0 \Rightarrow \text{(integration)}$$

$$\psi_1'\psi_2 - \psi_2'\psi_1 = \text{Const}$$

Since at $x \to \infty$ $\psi_1 \to 0$; $\psi_2 \to 0 \Rightarrow$

$$\text{Const} = 0 \Rightarrow \psi_1'\psi_2 - \psi_2'\psi_1 = 0 \Rightarrow$$

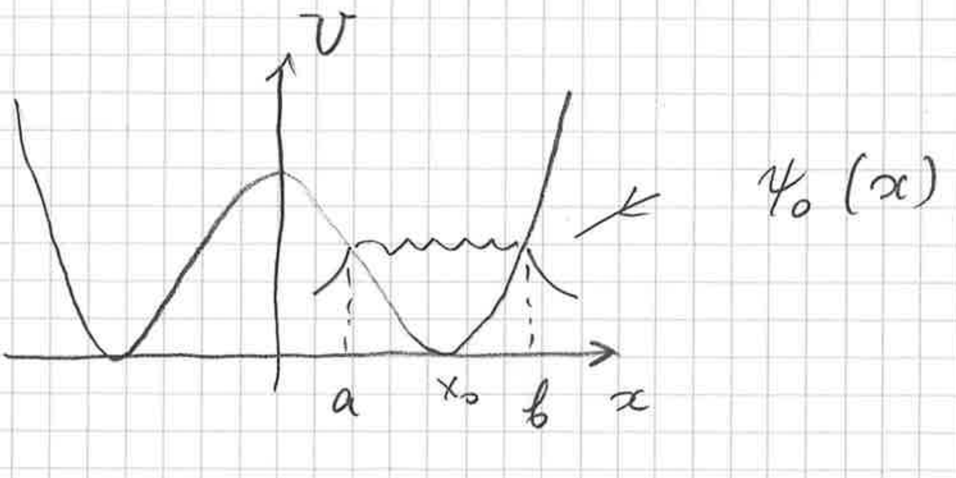
$$\frac{\psi_1'}{\psi_1} = \frac{\psi_2'}{\psi_2} \Rightarrow \psi_1 = \text{Const}\,\psi_2 \text{ —}$$

contradiction

So, in the double-well potential the degeneracy should be broken. Let us find the splitting between energy levels.

Generic considerations: our potential is symmetric with respect to parity transformation, $x \Rightarrow -x$ – it commutes with parity transformation. The lowest energy state must be symmetric under this transformation, while the 1st excited state must be antisymmetric. All other states can also be classified with respect to parity transformation: they are either even: $\psi(x) = \psi(-x)$ or odd, $\psi(x) = -\psi(-x)$.

Let us take a semiclassical wave function concentrated in the right well:



The functions of definite parity are:

even   odd

$$\psi_1 = \frac{1}{\sqrt{2}}\left(\psi_0(x) + \psi_0(-x)\right), \quad \psi_2 = \frac{1}{\sqrt{2}}\left(\psi_0(x) - \psi_0(-x)\right)$$

See Fig in page 7

Shrödinger equations for ψ_1 & ψ_2:

$$\psi_1'' + \frac{2m}{\hbar^2}(E_1 - U)\psi_1 = 0 \quad ; \text{ multiply by } \psi_2$$

$$\psi_2'' + \frac{2m}{\hbar^2}(E_2 - U)\psi_2 = 0 \quad ; \text{ multiply by } \psi_1$$

$$\left.\begin{aligned} &\psi_1''\psi_2 + \frac{2m}{\hbar^2}(E_1 - U)\psi_1\psi_2 = 0 \\ &\psi_2''\psi_1 + \frac{2m}{\hbar^2}(E_2 - U)\psi_2\psi_1 = 0 \end{aligned}\right] \text{ take a difference}$$

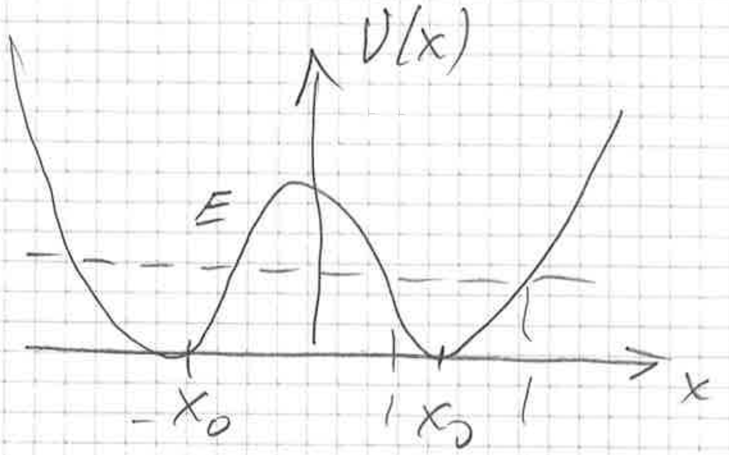
$$\psi_1''\psi_2 - \psi_2''\psi_1 + \frac{2m}{\hbar^2}(E_1 - E_2)\psi_1\psi_2 = 0$$

integrate from 0 to infinity:

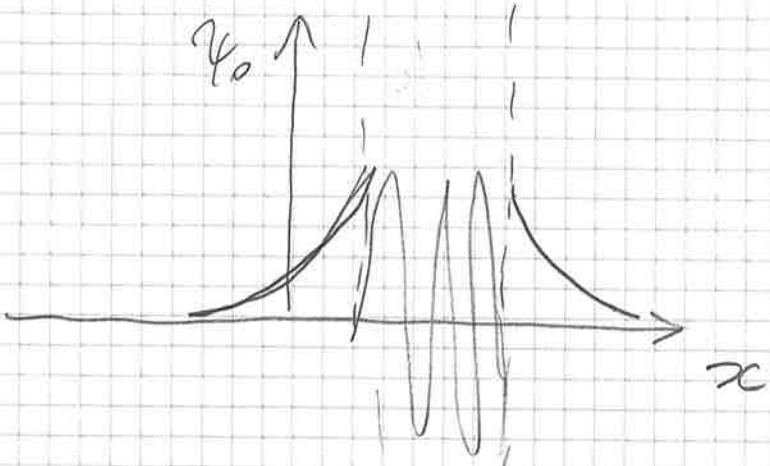
$$\left(\psi_1'\psi_2 - \psi_2'\psi_1\right)\Big|_0^\infty + \frac{2m}{\hbar^2}(E_1 - E_2)\int_0^\infty \psi_1\psi_2\,dx = 0$$

$\underbrace{\psi_1'\psi_2}_{= 0}$ since $\psi_2(0) = 0$

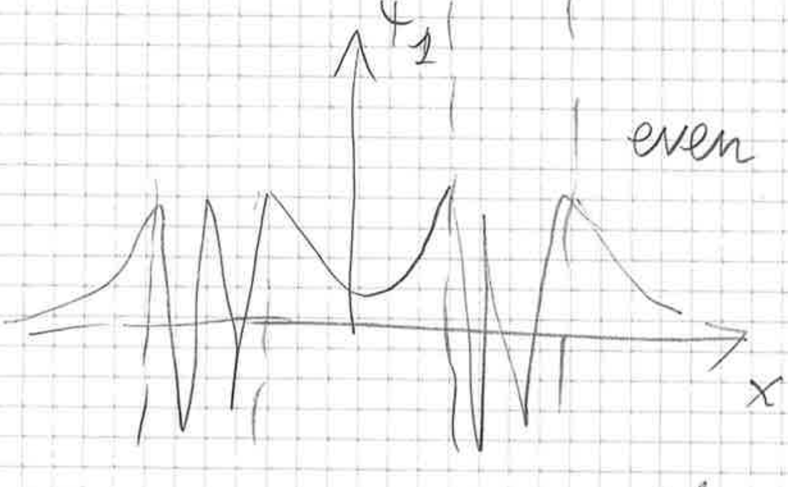
$\underbrace{\psi_2'\psi_1}_{=}$ $+\frac{1}{\sqrt{2}} \cdot 2\psi_0' \cdot \frac{1}{\sqrt{2}} 2\psi_0$



potential



concentrated in right well



even

$$\frac{1}{\sqrt{2}}\left(\psi_0(x)+\psi_0(-x)\right)$$

(smaller energy)

odd

$$\frac{1}{\sqrt{2}}\left(\psi_0(x)-\psi_0(-x)\right)$$

larger energy

$$\int_0^\infty \psi_1 \psi_2 \, dx = \int_0^\infty \frac{1}{\sqrt{2}} \psi_0(x) \cdot \frac{1}{\sqrt{2}} \psi_0(x) \, dx =$$

$= \frac{1}{2}$, since we normalize ψ_0 as:

$$\int_0^\infty |\psi_0|^2 dx = 1 \Rightarrow$$

$$2 \psi_0 \psi_0' \Big|_{x=0} + \frac{2m}{\hbar^2} (E_1 - E_2) \cdot \frac{1}{\cancel{2}} = 0$$

ψ_0 is a semiclassical wave function,

$$\psi = \begin{cases} \dfrac{C}{2\sqrt{p}} \exp\left(-\dfrac{1}{\hbar}\displaystyle\int_x^a |p|\, dx\right), & x < a \\ \dfrac{C}{\sqrt{p}} \cos\left[\dfrac{1}{\hbar}\displaystyle\int_a^x |p|\, dx - \dfrac{\pi}{4}\right], & a < x < b \end{cases}$$

To find $E_1 - E_2$, we also need to know

C - normalisation of the semiclassical

wave-function,

$C = 2\sqrt{\frac{m\omega}{2\pi}}$ (to be found at exercises)

page 7 excluded

So, $\psi(x) = \sqrt{\frac{m\omega}{2\pi p}} \exp\left(-\frac{1}{\hbar}\int_x^a |p|\, dx\right)$

Derivative: (only exponent should be differentiated, the derivative of the prefactor is small compared with the main term if semiclassical approximation is valid)

$$\psi_0' = \psi_0 \cdot \frac{1}{\hbar} \cdot p \Rightarrow$$

$$E_1 - E_2 = -\frac{2\hbar^2}{m} \cdot \left(\frac{m\omega}{2\pi p}\right)^{\frac{1}{2}\cdot 2} \frac{1}{\hbar} p \quad \exp\left[-\frac{1}{\hbar}\int_{-a}^{a} |p| dx\right]$$

$$= -\frac{\omega\hbar}{\pi} \exp\left[-\frac{1}{\hbar}\int_{-a}^{a} |p| dx\right]$$

At this point we end the chapter on semiclassical approximation. There are many other applications which we will not consider in these lectures

- multi-dimensional case, e.g. $D=3$.
- Spherically symmetric systems
- Semiclassical matrix elements
- Transition probabilities
- etc.

I think, however, that you have got the main ideas, and can study other cases by yourself, if this will be needed in your future work.

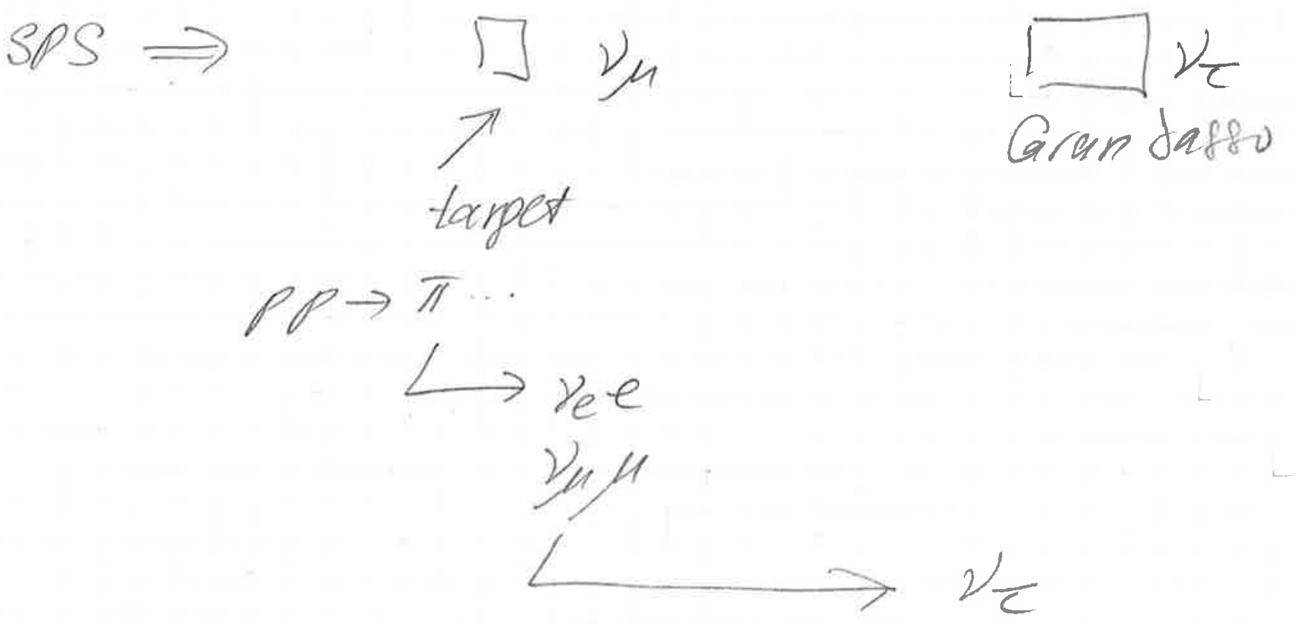
New topic: Quantum mechanics of (non-relativistic) collisions

Why collisions?

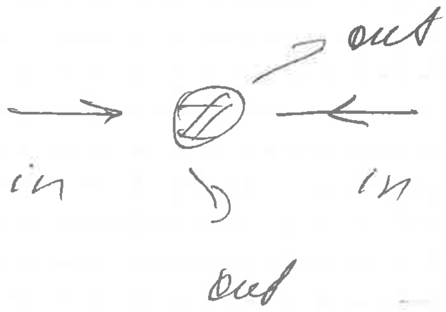
- study the properties of elementary particles, create new particles

(i) fixed target experiments

Example (recent):



(ii) Collider experiments



examples: LHC, $p+p$ @ 7 TeV initially
13 TeV now
FNAL, $p+\bar{p}$ (tevatron)
LEP, e^+e^-, etc

- Nuclear physics, nuclear fusion
- Statistical physics, properties of substances
- astrophysics, properties of stars

Plan:

1. Collisions in classical mechanics
   - Definition of cross-section
2. Scattering in Quantum mechanics
   - S-Matrix and Møller operators
   - cross-section and scattering amplitude
   - optical theorem
3. Methods for computing S-matrix
   - Green functions
   - Lippmann-Schwinger equation
   - Born series
4. Stationary scattering states

References:

T. Taylor: Scattering theory (available @ Google books)

Landau & Lifshitz volume III

1 Collisions in classical mechanics

Potential scattering: particle with mass m is moving in potential $V(\vec{x})$; $V(\vec{x}) \to 0$ at $x \to \infty$

← region of interaction

initial condition:

$$x(t) \to \vec{x}_{in} + \vec{v}_{in} \cdot t, \quad \text{for } t \to -\infty$$

final state:

$$x(t) \to \vec{x}_{out} + \vec{v}_{out} \cdot t, \quad t \to +\infty \qquad (*)$$

The problem of scattering theory:

find
$$\vec{x}_{out} = \vec{x}_{out}(\vec{x}_{in}, \vec{v}_{in})$$
$$\vec{v}_{out} = \vec{v}_{out}(\vec{x}_{in}, \vec{v}_{in})$$

Mathematically:

- find all trajectories of particles from the Newton equation

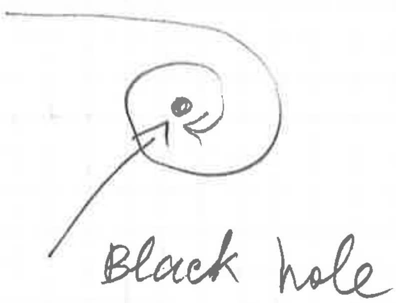
$$m\ddot{\vec{x}} = -\frac{\partial V}{\partial x}$$

- select trajectories that coincide with $t \to -\infty$ asymptotics
- determine $\vec{x}_{out}$ and $\vec{v}_{out}$

Remarks:

- not all trajectories have asymptotics (*), some of them can be periodic,

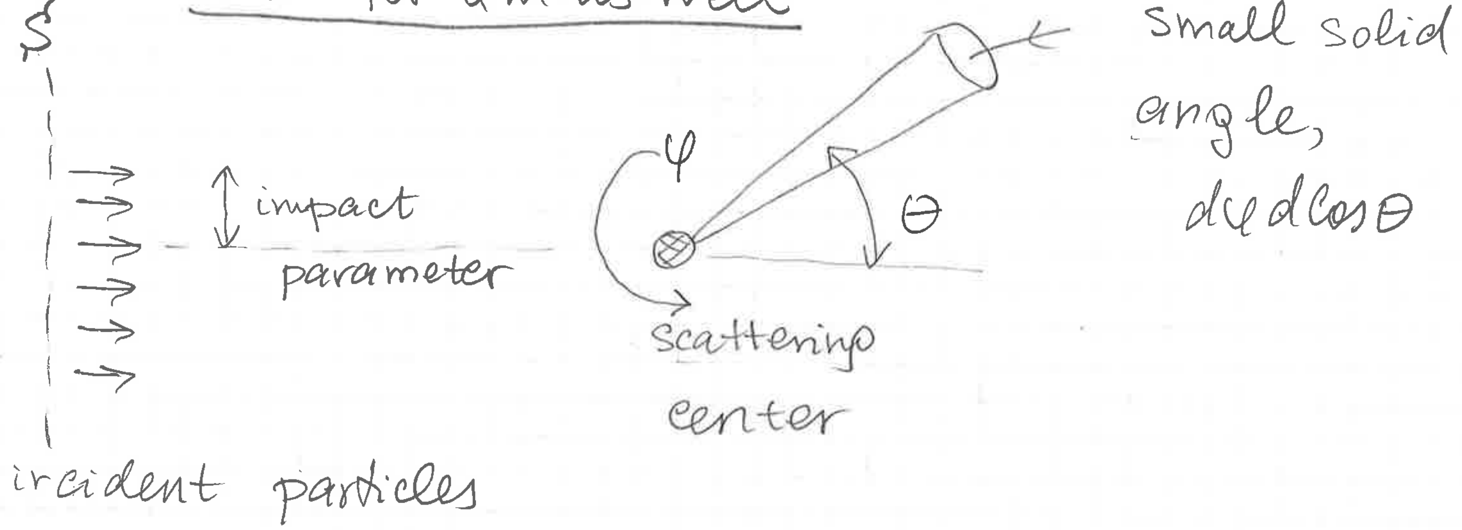
like for planets in Solar system, and some of them may not have outgoing trajectories,



Examples will be given at GR lectures (black holes)

After the problem of scattering-theory is solved, we can find the main quantity of scattering theory - the cross-section

Definition of cross-section valid for QM as well



Flux of incident particles is distributed homogeneously with density

$$n = \text{const} \ \frac{1}{cm^2 s}$$

(this is the number of particles crossing the plane S, per unit time and unit area).
All particles have the same momentum p.
Let $\frac{dN}{dt}$ is the number of particles going in direction θ, φ within the solid angle $d\Omega = d\cos\theta \, d\varphi$ per unite time

Then:
$$\frac{dN}{dt} = \underbrace{\frac{d\sigma}{d\Omega}}_{\text{definition of differential cross-section}} \cdot n \cdot d\Omega$$

Dimension of cross-section:

$$\left[\frac{dN}{dt}\right]: \frac{1}{s}\,;\quad [n]: \frac{1}{sm^2 s} \Rightarrow \left[\frac{d\sigma}{d\Omega}\right]: cm^2$$

Total cross-section:

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

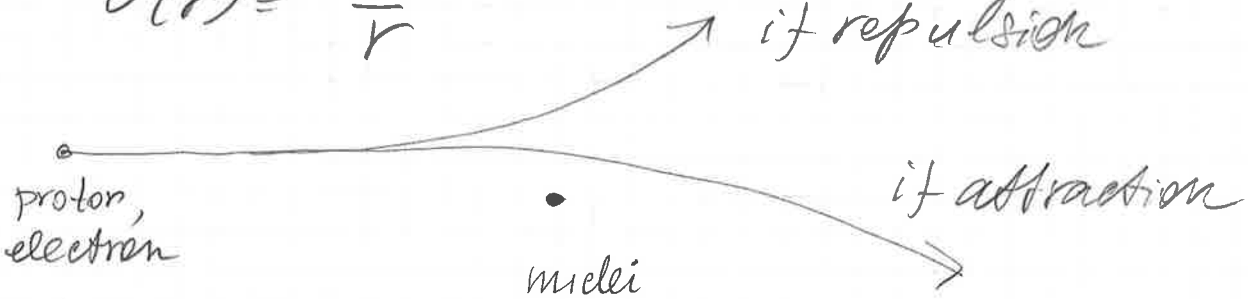
Example: scattering of point-like particles on a ball of radius r:

total cross-section:

$$\underline{\sigma = \pi r^2}$$

More complicated example, Reserford scattering; potential (Coulomb),

$$V(r) = \frac{\alpha}{r}$$



Result:

$$\frac{d\sigma}{d\Omega} = \left[\frac{\alpha}{2mv^2}\right]^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

where v is the initial velocity.

Derivation of the Reserford formula is left for exersise. Here are indications:

(i) trajectory of particle and the scattering center lie in the same plane. This is a consequence of angular momentum conservation in central potential.

(ii) For central problem, the best choice of coordinates is r and θ

(iii) Write angular momentum conservation in terms of r & θ

(iv) Write energy conservation in terms of r & θ

(v) express scattering angle via impact parameter

Remark: potential scattering and 2×2 scattering are equivalent.

Indeed, consider two particles with masses m_1 and m_2, and interacting with potential $V(|\vec{x}_1-\vec{x}_2|)$. This problem is reduced easily to the problem of potential scattering:

$$m_1 \ddot{\vec{x}}_1 = -\frac{\partial V}{\partial x_1} \; ; \quad m_2 \ddot{\vec{x}}_2 = -\frac{\partial V}{\partial x_2} \qquad (**)$$

Introduce relative distance

$$x = x_1 - x_2$$

and the position of the center of mass,

$$\vec{X} = \frac{m_1 \vec{x}_1 + m_2 \vec{X}_2}{m_1 + m_2}$$

Then $(**)$ are reduced to:

$$\ddot{\vec{X}} = 0 \Rightarrow \vec{X} = \vec{X}_0 + \vec{V}_0 t$$

$$m_{eff}\, \ddot{\vec{x}} = -\frac{\partial V}{\partial \vec{x}} \, , \quad m_{eff} = \frac{m_1 m_2}{m_1 + m_2}$$

Cross-section in different frames

The most used frames are:

- Laboratory frame: incident particle 1, initial momentum: p_1
  target particle 2, initial momentum $p_2 = 0$ (fixed target)
- Center of mass frame,
  total momentum $= 0$, $\vec{p}_1 + \vec{p}_2 = 0$,
  $\vec{X} = 0$ (LHC, LEP, FNAL)

Asymmetric kinematics: BELLE (B-factory) 8 GeV e^-, 3.5 GeV e^+

Relation between the cross-sections in these two frames (lab and Center of mass)

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \frac{\left(1 + 2\lambda \cos\theta_{cm} + \lambda^2\right)^{3/2}}{1 + \lambda \cos\theta_{cm}}$$

where $\lambda = m_1/m_2$, and θ_{cm} is the scattering angle in the center-of-mass frame. The proof is left for exercise

Important characteristics of accelerator: luminocity $\mathcal{L}$. Definition of $\mathcal{L}$.

Number of events we are looking at [e.g. $p+p \to H +$ anything],
Higgs boson

is equal to

$$\frac{dN}{dt} = \mathcal{L} \cdot \sigma$$

cross-section for the specified channel

Dimensions:

$$\left[\frac{dN}{dt}\right] : \frac{1}{s} ; \quad [\sigma] : cm^2 \Rightarrow [\mathcal{L}] = \frac{1}{cm^2 \cdot s}$$

Design luminocity of the LHC:

$$10^{34} \frac{1}{cm^2 s}$$

Integrated luminocity:

$$\int \mathcal{L} dt, \text{ dimension: } \frac{1}{cm^2}$$