Lecture # 10

Quantum physics 3

Main points of # 11

(i) S - matrix coinsides with evolution operator in Dirac (interaction) picture of Quantum mechanics

(ii) S - matrix is a unitary operator, provided certain conditions on potential are satisfied

(iii) Møller operators are isometric, if there are bound states, and unitary, if there are none

(iv) Proved that $H\Omega_{\pm} = \Omega_{\pm} H_0$, and that $[H_0, S] = 0$

(v) introduced scattering amplitude,

$$\langle \bar{p}' | S | \bar{p} \rangle = \delta^3(\bar{p}' - \bar{p}) + \frac{i}{2\pi m} \delta(E_p - E_{p'}) f(\vec{p}' \leftarrow \vec{p})$$

1. Optical theorem
2. Demonstration that cross-section is the square of the scattering amplitude
3. Greens functions and the scattering amplitude
4. Analytic properties of Green's functions

From now on, we will use the system of units with $\hbar = 1$

Optical theorem reads:

$$\operatorname{Im} f(\vec{p} \leftarrow \vec{p}) = \frac{|p|}{4\pi} \sigma(p)$$

total cross-section

The proof of optical theorem:

$S = 1 + R$, $S^+ = 1 + R^+$, $SS^+ = 1 \Rightarrow$

$$R + R^+ = -RR^+$$ matrix element $\langle p'| \dots |p\rangle$

$$\langle p'|R|p\rangle + \langle p|R|p'\rangle^* = -\int d^3p'' \langle p'|R^+|p''\rangle\langle p''|R|p\rangle$$

with $\langle p'|R|p\rangle = \frac{i}{2\pi m}\, \delta(E_p - E_{p'})\, f(\bar{p}' \leftarrow \bar{p})$:

$$\frac{i}{2\pi m}\delta(E_p - E_{p'})\left[f(p' \leftarrow p) - f^*(p \leftarrow p')\right] =$$

$$= -\left(\frac{i}{2\pi m}\right)\left(-\frac{i}{2\pi m}\right)\int d^3p''\, \delta(E_{p'} - E_{p''}) \cdot \delta(E_{p''} - E_p) \cdot$$

$$\cdot f^*(p'' \leftarrow p')\, f(p'' \leftarrow p)$$

Since $\delta(E_{p'} - E_{p''})\,\delta(E_{p''} - E_p) = \delta(E_p - E_{p'})\,\delta(E_p - E_{p''})$

we get:

$$f(\bar{p}' \leftarrow \bar{p}) - f^*(\bar{p} \leftarrow \bar{p}') = \left(\frac{i}{2\pi m}\right)\int d^3p''\, \delta(E_p - E_{p''}) \cdot$$

$$\cdot f^*(\bar{p}'' \leftarrow \bar{p}')\, f(\bar{p}'' \leftarrow \bar{p})$$

Now, for $\bar{p}' = \bar{p}$ we get:

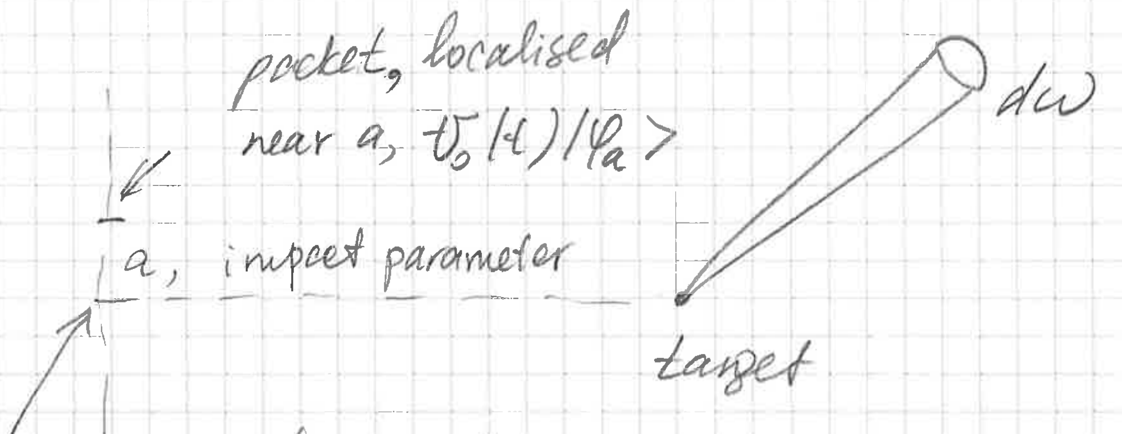
$$2i\, Im\, f(\bar{p} \leftarrow \bar{p}) = \frac{i}{2\pi m}\int d\Omega\, p''^2 dp''\, \underbrace{\delta(E_p - E_{p''})}_{\frac{2m}{2p}\delta(p - p'')} \cdot |f(p'' \leftarrow p)|^2$$

$$= \frac{i}{2\pi m}\,\frac{m}{p}\,p^2 \int d\Omega\, \frac{d\sigma}{d\Omega} =$$

$$= \frac{i}{2\pi} \cdot p\,\sigma \Rightarrow Im\, f(\vec{p} \leftarrow \vec{p}) = \frac{|p|}{4\pi}\sigma(p)$$

Im. part of the scattering amplitude is total cross-section.

Cross-section and scattering amplitude



packet, localised near $a = 0$,

$$U_0(t)\,|\varphi\rangle$$

$|\varphi\rangle$: normalised wave packet, with momentum near $\vec{p}_0$, and impact factor near $a = 0$

$|\varphi_a\rangle = e^{-i\vec{p}\vec{a}}\,|\varphi\rangle$: translation by impact factor $\vec{a}$, translation operator $e^{-i\vec{p}\vec{a}}$

Probability to detect particle in direction $d\Omega$, originating from $|\psi_{in}\rangle$:

$$w(d\Omega \leftarrow \psi_{in}) = d\Omega \int p^2 dp\, |\psi_{out}(\vec{p})|^2$$

total number of events:

$$N = \int d^2a \; n_{in}\, w(d\Omega \leftarrow \varphi_a)$$

integration over plane

uniform distribution

So,

$$\frac{d\sigma}{d\Omega} d\Omega = \int d^2 a \, w(d\Omega \leftarrow \varphi_a)$$

We will find that the cross-section does not depend on details of the wave-function φ: what is important, that it is peaked near $\vec{p}_0$. Let us find ψ_{out}:

$$\psi_{out} = \int d^3 p' \langle p|S|p'\rangle \psi_{in}(p') = \qquad (*)$$

$$= \psi_{in}(p) + \frac{i}{2\pi m} \int d^3 p' \, \delta(E_p - E_{p'}) f(\bar{p} \leftarrow \bar{p}') \cdot$$

$$\cdot \psi_{in}(p') \, ; \qquad \psi_{in}(p') = e^{-i\bar{a}\bar{p}'} |\varphi\rangle$$

Let us take $\vec{p}$ not in direction of $\vec{p}_0$ – (exclude forward direction). Then the first term in $(*)$ does not contribute.

So,

$$\psi_{out} = \frac{i}{2\pi m} \int d^3 p' \, \delta(E_p - E_{p'}) f(\bar{p} \leftarrow \bar{p}') \cdot$$

$$\cdot e^{-i\bar{a}\bar{p}'} \varphi(p')$$

and

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{d\Omega}{(2\pi m)^2} \int d^2a \cdot \int p^2 dp \times$$

$$\int d^3p' \, \delta(E_p - E_{p'}) \, f(\vec{p} \leftarrow \vec{p}\,') e^{-i\vec{a}\vec{p}\,'} \varphi(p') \times$$

$$\int d^3p'' \, \delta(E_p - E_{p''}) \, f^*(\vec{p} \leftarrow \vec{p}\,'') e^{i\vec{a}\vec{p}\,''} \varphi^*(p'')$$

Integrals can be taken with the help of δ-functions:

$$\int d^2a \, e^{-(i\vec{a}\vec{p}\,' - i\vec{a}\vec{p}\,'')} = (2\pi)^2 \delta^2(p_\perp' - p_\perp'');$$

$$\delta^2(p_\perp' - p_\perp'') \, \delta(E_p - E_{p''}) \, \delta(E_p - E_{p'}) =$$

$$= \delta(E_p - E_{p'}) \, \delta\left((p_\parallel')^2 - (p_\parallel'')^2\right) \cdot 2m \cdot \delta^2(p_\perp' - p_\perp'') =$$

$$= \delta(E_p - E_{p'}) \cdot 2m \cdot \frac{1}{2|p_\parallel|} \cdot \delta^3(\vec{p}\,' - \vec{p}\,'') \Rightarrow$$

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{d\Omega}{(2\pi m)^2} \cdot (2\pi)^2 \cdot \int p^2 dp \times$$

$$\int d^3p' d^3p'' \, \delta(E_p - E_{p'}) \, \varphi(p') \varphi^*(p'') \, f(\vec{p} \leftarrow \vec{p}\,') f^*(\vec{p} \leftarrow p'')$$

$$\cdot \frac{m}{|p_\parallel|} \delta^3(\vec{p}\,' - \vec{p}\,'') =$$

$$= \frac{d\Omega}{m} \int p^2 dp \int \frac{d^3p'}{p_{\parallel}} \delta(E_p - E_{p'}) \cdot |\varphi(p')|^2 |f(\bar{p} \leftarrow \bar{p}')|^2$$

$$= \frac{d\Omega}{m} \int p^2 dp \int \frac{d^3p'}{p_{\parallel}} \frac{m}{p} \delta(|\bar{p}| - |\bar{p}'|) \cdot |\varphi(p')|^2 \cdot |f|^2 =$$

$$= d\Omega \int d^3p' |\varphi(p')|^2 |f(\bar{p} \leftarrow p')|^2 =$$

$$= |f(\bar{p} \leftarrow \bar{p}_0)|^2$$

used the fact that $\varphi(p')$ is peaked near $\vec{p}_0$, and that $\varphi(p')$ is normalised to 1.

QED: $\frac{d\sigma}{d\Omega} = |f(\bar{p} \leftarrow \bar{p}_0)|^2$

Now, our aim will be to construct the formalism for computation of the scattering amplitude.

Green's functions and the scattering amplitude.

Once more,

$$|\Psi_0\rangle = \Omega_+ |\Psi_{in}\rangle; \quad |\Psi_0\rangle = \Omega_- |\Psi_{out}\rangle;$$

$$\Omega_\pm = \lim_{t \to \mp\infty} U^+(t) U^0(t)$$

Useful representation of $U^+(t)U^0(t)$:

$$U^+(t)U_0(t) = 1 + \int_0^t dt' \frac{d}{dt'}\left(U^+(t')U_0(t')\right)$$

Now: $\frac{d}{dt}\left(e^{iHt} e^{-iH_0 t}\right) =$

$$iH e^{iHt} e^{-iH_0 t} - i e^{iHt} e^{-iH_0 t} H_0 =$$

$$= i e^{iHt}(H - H_0) e^{-iH_0 t} = i U^+(t) V U_0(t) \Rightarrow$$

$$\Psi_0 = |\Psi_{out}\rangle + i \int_0^\infty d\tau\, U^+(\tau) V U_0(\tau) |\Psi_{out}\rangle$$

$$= |\Psi_{out}\rangle + i \lim_{\varepsilon \to 0} \int_0^\infty d\tau\, e^{-\varepsilon\tau} U^+(\tau) V U_0(\tau) |\Psi_{out}\rangle$$

$$=$$

$$= |\Psi_{out}\rangle + i\int d^3p \int_0^\infty d\tau\, e^{-\varepsilon\tau + iH\tau}\, V e^{-iE_p\tau} |p\rangle\langle p|\Psi_{out}\rangle$$

$$= |\Psi_{out}\rangle + \int d^3p \frac{1}{E_p - H - i\varepsilon} V |p\rangle\langle p|\Psi_{out}\rangle$$

The object

$G(z) = \frac{1}{z-H}$, where z is arbitrary complex number is called Green's function.

We can write:

$$|\Psi_0\rangle = |\Psi_{out}\rangle + \int d^3p\, G(E_p - i\varepsilon) V |p\rangle\langle p|\Psi_{out}\rangle$$

The same for Ψ_{in}:

$$|\Psi_0\rangle = |\Psi_{in}\rangle + \int d^3p\, G(E_p + i\varepsilon) V |p\rangle\langle p|\Psi_{in}\rangle$$

Or, in terms of $\Omega_\pm$:

$$\boxed{\Omega_\pm = 1 + \int d^3p\, G(E_p \pm i\varepsilon) V |p\rangle\langle p|}$$

Important: explicite seperation of interaction associated with V.

Origin of the name (Green's function)

Let us consider identity for H_0 and G_0:

$$(z - H_0)\, G_0(z) = 1\,, \quad G_0 \equiv \frac{1}{z - H_0} \quad (*)$$

(exactly the same can be done for H and $G(z)$).

Use x-representation for $(*)$

$\langle x|(z - H_0)G_0|x'\rangle = \delta(x - x')$, meaning that

$$\int dx'' \langle x|(z - H_0)|x''\rangle\langle x''|G_0|x'\rangle = \delta(x - x')$$

$$= \int dx'' \left[z\,\delta(x - x'') + \frac{\nabla^2}{2m}\delta(x - x'') \right] \langle x''|G_0|x'\rangle$$

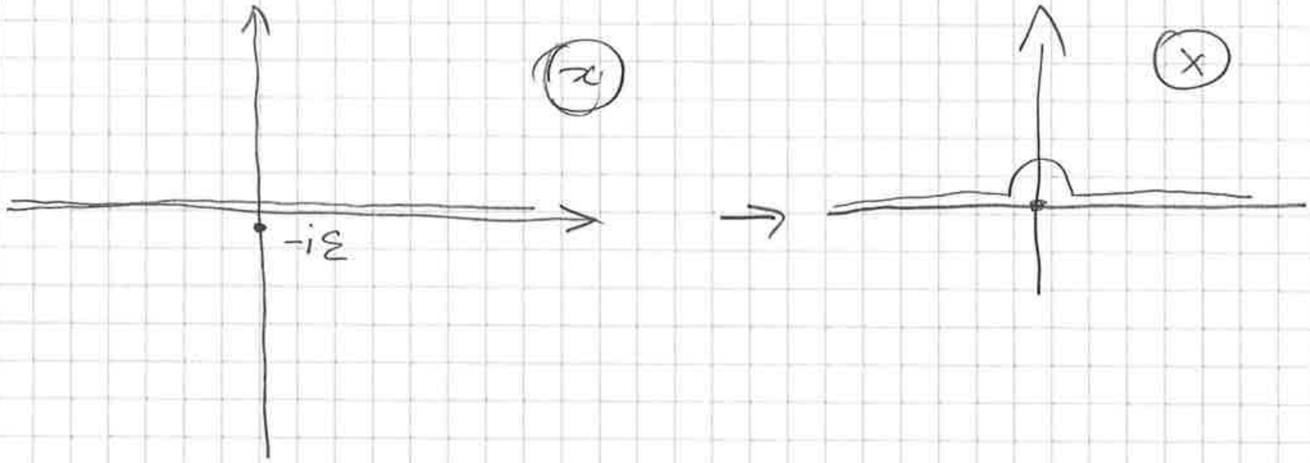
$$= \left(z + \frac{\nabla_x^2}{2m} \right) \langle x|G_0|x'\rangle = \delta(x - x')$$

So, $\langle x|G_0|x'\rangle$ is the Green's function for operator $z + \frac{\nabla^2}{2m}$

$\langle x|G|x'\rangle$ is the Green's function for operator $\left[z + \frac{\nabla^2}{2m} - V(x) \right] \langle x|G|x'\rangle = \delta(x - x')$

Useful formula:

$$\frac{1}{x+i\varepsilon} = \mathcal{P}\left(\frac{1}{x}\right) - \pi i \delta(x)$$



$\mathcal{P}$ - principal value:

$$\mathcal{P}\int \frac{1}{x}\varphi(x)\,dx \equiv \lim_{\varepsilon \to 0}\left[\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)\,dx}{x} + \int_{\varepsilon}^{\infty} \frac{\varphi(x)\,dx}{x}\right]$$