Lecture # 11

Quantum physics 3

Main points of #10

- Proved optic theorem
- Demonstrated that $\frac{d\sigma}{d\Omega} = |f|^2$
- introduced Green's functions

$G(z) = \frac{1}{z-H}$ and expressed Møller

operators via G

Today:

- analytic structure of G
- Lippmann-Schwinger equation
- S-matrix via Greens functions
- T-matrix, Born approximation

Analytic structure of $G(z)$ as a function of complex variable z

Let the Hamiltonian H has some bound states $|n\rangle$, $n = 1 .. N$, and then continuous spectrum of states parametrised by E_p:

$$H\,\Omega_+|\vec{p}\rangle = E_p\,\Omega_+|\vec{p}\rangle \equiv E_{\bar{p}}\,|\vec{q}\rangle;$$

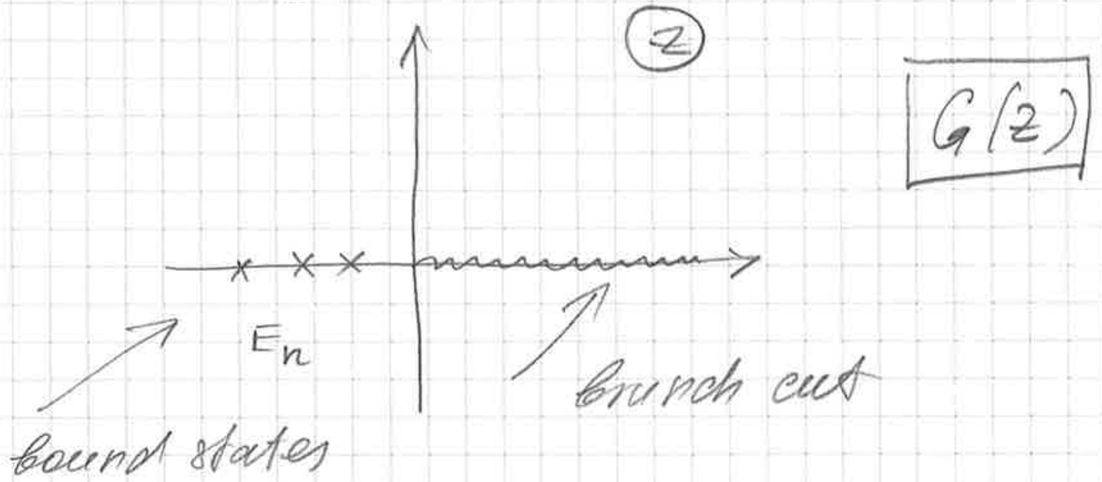
$$\vec{q} = \Omega_+|\vec{p}\rangle \quad ; \bar{E}_q = \bar{E}_p = \frac{q^2}{2m}$$

Then $G(z)$ can be written with the use of eigenvectors of H as:

$$G(z) = \frac{1}{z-H}\underbrace{\left\{\sum_1^N |n\rangle\langle n| + \int d^3q\, |\vec{q}\rangle\langle\vec{q}|\right\}}_{\mathbb{1}} =$$

$$= \sum \frac{|n\rangle\langle n|}{z-E_n} + \int d^3q\,\frac{|\vec{q}\rangle\langle\vec{q}|}{z-E_q}$$

This means that $G(z)$ is an analytic function of z everywhere on complex plain with exception of poles at $z = E_n$ and a brunch cut on real axis, $Re\, z > 0$, $Im\, z = 0$, associated with scattering states.



Analytic structure of $G_0(z)$:
no bound states, just scattering states, brunch cut at $Re\, z > 0$, $Im\, z = 0$

The residues of $G(z)$ at poles are $|n\rangle\langle n|$, giving the wave-functions of the bound states.

If we know $G(z)$, we can solve all QM problems: bound states, scattering, etc.

The Green's function $G(z)$ satisfies important equation, known as Lippmann-Schwinger equation.

To get it, let us introduce Green's function associated with H_0, $G_0(z)$:

$$G_0(z) = \frac{1}{z - H_0}$$

and use the obvious identity

$$A^{-1} = B^{-1} + B^{-1}(B-A)A^{-1},$$

taking $A = z - H$; $B = z - H_0$:

$$\boxed{G(z) = G_0(z) + G_0(z)\, V\, G(z)}$$

or, taking $A = z_0 - H$, $B = z - H$:

$$G_0 = G + G(-V)\, G_0,$$

or, the same,

$$\boxed{G = G_0 + G V G_0}$$

S-matrix via Green's functions:

$$S = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} e^{iH_0 t} e^{-iHt} e^{iHt'} e^{-iH_0 t'} =$$

(take $t' = -t$)

$$= \lim_{t\to+\infty} \left[e^{iH_0 t} e^{-2iHt} e^{iH_0 t} \right]$$

The same trick:

$$\frac{d}{dt}[\ldots] = -i\left[e^{iH_0 t} V e^{-2iHt} e^{iH_0 t} + e^{iH_0 t} e^{-2iHt} V e^{iH_0 t} \right];$$

$$\langle p'|S|p\rangle = \delta(\vec{p}-\vec{p}\,') -$$

$$-i \lim_{\varepsilon\to+0} \int_0^\infty dt \, \langle p'| V e^{-\varepsilon t + iE_{p'}t + iE_p t - 2iHt} + e^{-\varepsilon t + iE_{p'}t + iE_p t - 2iHt} V |p\rangle$$

$$= \delta(\vec{p}-\vec{p}\,') + \frac{1}{2} \langle p'| \left\{ V G\left(\frac{E_{p'}+E_p}{2} + i\varepsilon\right) + G\left(\frac{E_{p'}+E_p}{2} + i\varepsilon\right) V \right\} |p\rangle \qquad (**)$$

The combination in curly brackets on page 1 is an important object in scattering theory, related to "T"-matrix:

Definition of T:

$$T(z) = V + VG(z)V,$$

where V is the potential, and G is the Green's function we just defined.

Relations between T, G, G_0, and V:

$$G_0 T = G_0 V + G_0 V G V = (G_0 + G_0 V G) V =$$

$$= GV \quad \text{(use of Lippmann-Schwinger eq)}$$

$$\boxed{G_0 T = G V}$$

also, $TG_0 = (V + VGV)G_0 = V(G_0 + GVG_0)$

$$= VG$$

$$\boxed{TG_0 = VG}$$

also, $\boxed{G = G^0 + G^0 T G^0}$,

since $G_0 + G_0 T G_0 = G_0 + G V G_0 = G$,

as a consequence of L-S. equation

Lippmann-Schwinger equation for T:

$$T = V + V\underbrace{G V}_{G_0 T} = V + V G_0 T$$

$$\boxed{T = V + V G_0 T}$$

Now, we are ready to simplify expression in curly brackets in (**):

$$\{VG + GV\} = \{T G_0 + G_0 T\} \Rightarrow$$

$$\langle p' | \{ \quad \} | p \rangle = \langle p' | T\left(\frac{E_p + E_{p'}}{2} + i\varepsilon\right) | p \rangle \cdot$$

$$\cdot \left[\frac{1}{\frac{E_{p'} + E_p}{2} + i\varepsilon - E_p} + \frac{1}{\frac{E_{p'} + E_p}{2} + i\varepsilon - E_{p'}} \right] =$$

$$= \langle p' | T | p \rangle \left[\frac{2}{E_{p'} - E_p + i\varepsilon} + \frac{2}{E_p - E_{p'} + i\varepsilon} \right] =$$

Now, use $\frac{1}{x + i\varepsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$

principal value

$$= \langle p'|T|p\rangle \cdot (-4\pi i)\, \delta(E_p - E_{p'})$$

Therefore,

$$\langle p'|S|p\rangle = \delta(\vec{p}-\vec{p}\,') - 2\pi i\, \delta(E_p - E_{p'}) \cdot$$

$$\cdot \langle p|T(E_p + i\varepsilon)|p'\rangle$$

So, T- matrix „on shell"
(on shell means that the argument z
is ~~equal~~ to the physical value E_p up
to small $i\varepsilon$) coinsides with the
scattering amplitude up to a coefficient,

$$t(\bar{p}\,' \leftarrow \bar{p}) \equiv \langle p'|T(E_p + i\varepsilon)|p\rangle = -\frac{f(\bar{p}\,' \leftarrow \bar{p})}{(2\pi)^2 \cdot m}$$

Now, we are ready for construction
of perturbation theory for
the scattering amplitude

Perturbation theory for amplitudes

It is quite difficult to find S-matrix exactly. One of the methods is to use perturbation theory.

Suppose the interaction is small,

$$H = H_0 + \lambda V, \quad \lambda \ll 1$$

"Formal solution" of L-Sh-equation:

$$T = \lambda V + \lambda V G_0 T;$$

$$(1 - \lambda V G_0) T = \lambda V;$$

$$T = (1 - \lambda V G_0)^{-1} \cdot \lambda V$$

Expansion:

1st order (Born approximation)

$$T = \lambda V$$

2nd order:

$$T = \lambda V + \lambda^2 V G_0 V$$

etc.

Scattering amplitude in the Born approximation:

$$f(\vec{p}\,' \leftarrow \vec{p}) = -(2\pi)^2 m \langle p' | V | p \rangle =$$

(put $a = 1$, i.e. absorb it to V)

$$= -(2\pi)^2 m \int \frac{d^3x}{(2\pi)^3} e^{-i\bar{p}'\bar{x}} V(x) e^{+i\bar{p}\bar{x}} =$$

$$= -\frac{m}{(2\pi)} \int d^3x \, e^{-i\bar{q}\bar{x}} V(x)$$

The quantity $\vec{q} = \bar{p}' - \bar{p}$ is called momentum transfer.

Computation of the second order of perturbation theory:

$$T^{(1)} = V$$

$$T^{(2)} = V G_0 V \; ;$$

$$\langle p' | T^{(2)} | p \rangle = \int d^3p'' \langle p' | V | p'' \rangle \langle p'' | G_0 V | p' \rangle$$

$$= \int d^3p'' \langle p' | V | p'' \rangle \frac{1}{\frac{E_{p'} + E_p}{2} + i\varepsilon - E_{p''}} \langle p'' | V | p' \rangle$$

This was already computed, Fourier transform of interaction potential

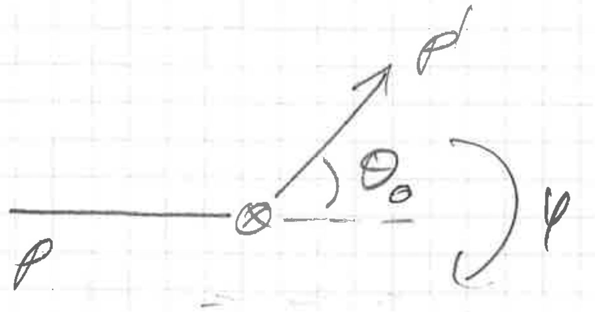
Example: The Ukawa potential

(approximation to nuclear forces)

$$V(r) = \frac{\alpha}{r} e^{-\mu r}$$

(it satisfy our requirements for existence and unitarity of the S-matrix)

In the limit $\mu \to 0$ we will get the Coulomb potential



$|\bar{p}| = |\bar{p}'|$ – energy conservation

$$f(\vec{p}\,' - \vec{p}) = -\frac{m}{2\pi}\int d^3x\, e^{-i\bar{q}\bar{x}}\,\frac{\alpha}{|x|}e^{-\mu|x|} =$$

(spherical coordinates) =

$$= -\frac{m}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi}\sin\theta\, d\theta \int_0^{\infty} r^2 dr\, e^{-iqr\cos\theta}\,\frac{\alpha}{r}e^{-\mu r}$$

elementary integral, left for exersises

$$= -\frac{2m\alpha}{q^2+\mu^2}$$

momentum transfer:

$$q^2 = (\bar{p}-\bar{p}')^2 = p^2 + p'^2 - 2|p|^2\cos\theta_0 =$$

$$= 4p^2\sin^2\frac{\theta_0}{2}$$

So, $$\frac{d\sigma}{d\Omega} = \frac{(2m\alpha)^2}{\left(\mu^2 + 4p^2\sin^2\frac{\theta_0}{2}\right)^2}$$

In the limit $\mu \to 0$, this coinsides with the classical cross-section for Coulomb potential!