Lecture # 14

- Discussed different attempts to make quantum mechanics relativistic

- Derived Dirac equation,

$$-\frac{1}{i}\frac{\partial\psi}{\partial t} = H\psi; \quad H = \alpha_i p_i + \beta m,$$

$$\{\alpha_i \alpha_j\} = 2\delta_{ij}, \ \beta^2 = 1, \ \{\alpha_i, \beta\} = 0,$$

$$\alpha_i^+ = \alpha_i, \ \beta = \beta^+ \quad - 4\times 4 \text{ matrices}$$

- Discussed Dirac equation in external EM field, $p^\mu \to p^\mu - \frac{e}{c}A^\mu$

- introduced $\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix}$, and then $\tilde{\varphi} = e^{-imt}\varphi$, $\tilde{\chi} = e^{-imt}\chi$, and got equations for them:

$$-\frac{1}{i}\frac{\partial\varphi}{\partial t} = \vec{\sigma}\vec{\pi}\chi + e\Phi\varphi$$

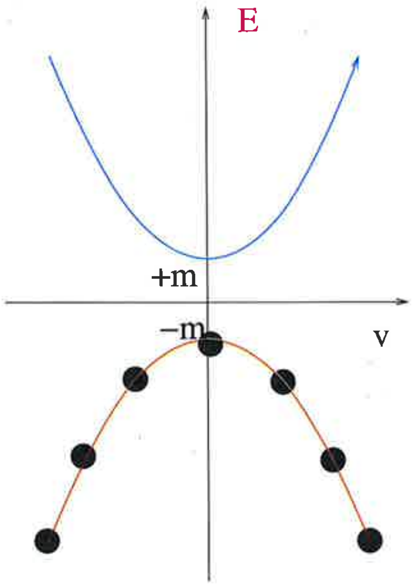
$$-\frac{1}{i}\frac{\partial\chi}{\partial t} = \vec{\sigma}\vec{\pi}\varphi + e\Phi\chi - 2m\chi$$

Plan: finish with Pauli, relativistic corrections to Pauli equation, correct formulation of relat. QM

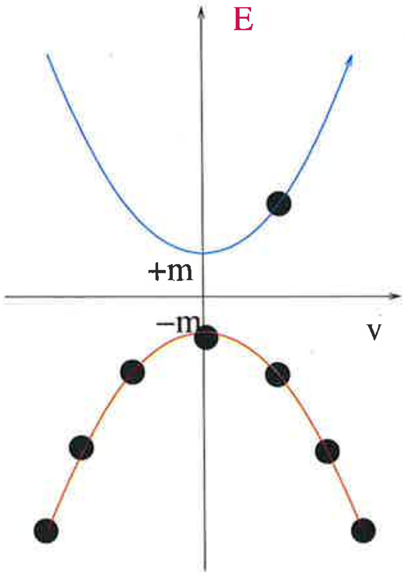
1a

Dirac proposal: Let us fill all negative energy levels by electrons and call this state a vacuum. Then it is stable due to the Pauli principle!

Pauli principle: no more than one fermion for a quantum state.

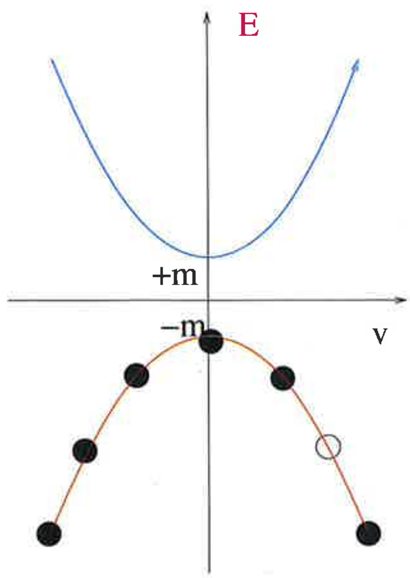


vacuum



particle

Holes in the Dirac vacuum (sea) are particles with positive energies and positive electric charge



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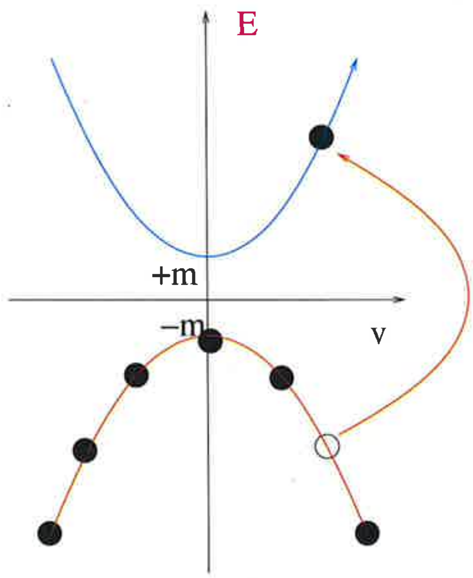
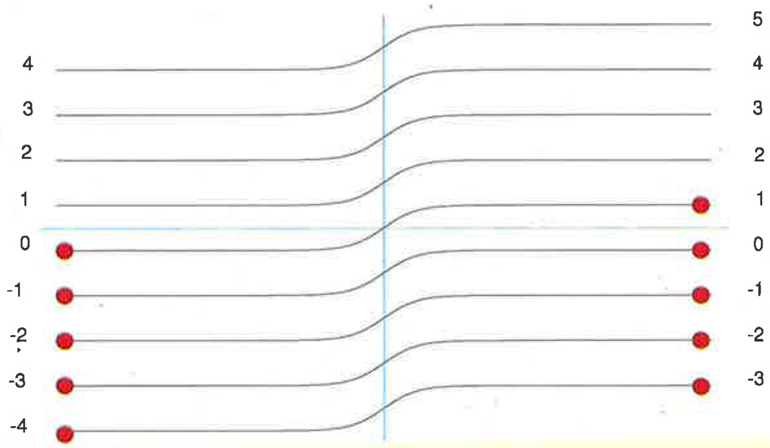


pair creation

Another possibility – Fermion number non-conservation, realised in the Standard Model

Adiabatic change from, A_1=0 non-zero $A_1 = 2\pi/eL$:



Process:

$$\text{bosons} \rightarrow \text{bosons} + 6q + 3\ \text{leptons},$$

e.g. $D = (pn) \rightarrow \bar{p} + e^+ + \mu^+ + \bar{\nu}_\tau$

(2)

Let us analyse second equation.

Since $e\Phi << 2mc^2$ [weak fields - valid in the atom: potential energy of electron in the atom ~ binding energy ~ 13 eV << $mc^2 = 5 \cdot 10^5$ eV]

we can drop the term $e\phi\chi$.

Rewrite second equation as

$$\chi = \frac{c\vec{\sigma}\vec{\pi} \cdot \varphi}{2mc^2} + \frac{\hbar}{i}\frac{1}{2mc^2}\frac{\partial\chi}{\partial t}$$

and solve it by itterations, assuming that term with $\dot{\chi}$ is small:

$$\chi_0 \approx \frac{c\vec{\sigma}\vec{\pi}}{2mc^2}\varphi \qquad \text{(zero approximation)}$$

$$\chi_1 \approx \frac{c\vec{\sigma}\vec{\pi}\varphi}{2mc^2} + \underbrace{\frac{\hbar}{i}\frac{1}{2mc^2}\frac{\partial}{\partial t}\left[\frac{c\vec{\sigma}\vec{\pi}}{2mc^2}\cdot\varphi\right]}$$

suppressed in comparison with the firs term by extra mc^2!

So, we will keep the first term only.

χ_0 is << than φ. χ is called „small" component, and φ is called „large" component.

insert this solution to equation for the first line in (xx):

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\varphi = c\vec{\sigma}\vec{\pi}\cdot\underbrace{\chi}_{\frac{\vec{\sigma}\vec{\pi}\varphi}{2mc}} + e\Phi\cdot\varphi =$$

$$= \left(\frac{\vec{\sigma}\vec{\pi}\;\vec{\sigma}\cdot\vec{\pi}}{2m} + e\Phi\right)\varphi$$

Transform the first part:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\,\vec{a}\wedge\vec{b}$$

for our case:

$$(\vec{\sigma}\vec{\pi})(\vec{\sigma}\vec{\pi}) = \vec{\pi}^2 + i\sigma\underbrace{\vec{\pi}\wedge\vec{\pi}} = \vec{\pi}^2 - \frac{e\hbar}{c}\vec{\sigma}\cdot\vec{B}$$

not zero, as $\vec{p}$ does not commute with $\vec{A}$

At last,

$$-\frac{\hbar}{i}\frac{\partial\varphi}{\partial t} = \left[\frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{e\hbar}{2mc}\vec{\sigma}\vec{B} + e\Phi\right]\varphi -$$

exactly Pauli equation.

Two components of φ correspond to two degrees of freedom for electron.

for weak (uniform) magnetic fields, $\vec{B} = \mathrm{rot}\,\vec{A}$,

$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ we can expand with respect to $\vec{A}$ and get:

orbital moment, $L = \vec{r}\wedge\vec{p}$

$$-\frac{\hbar}{i}\frac{\partial\varphi}{\partial t} = \left[\frac{p^2}{2m} - \frac{e}{2mc}(\vec{L}+2\vec{S})B\right]\varphi,$$

leading to magnetic moment of electron

$$\mu = \frac{e}{2mc}$$

Relativistic corrections to Pauli equation: next order in expansion

$$\mathcal{H} = \mathcal{H}_{Pauli} + V_1 + V_2 + V_3$$

$$V_1 = -\frac{1}{2mc^2}\frac{(p^2)^2}{4m^2}$$ (due to relativistic energy)

$$V_2 = +\frac{1}{2mc^2}\frac{1}{r}\frac{dV}{dr}\vec{S}\cdot\vec{L}, \quad V = -\frac{Ze^2}{r}$$ – Spin-orbital

$$V_3 = \frac{\hbar^2\pi}{2m^2c^2}Z\delta(x)\cdot e^2$$ – contact interaction

One can go even further and find higher order corrections. These results are to be used to find fine structure of the energy levels of hydrogen atom.

Region of applicability of perturbation theory:

$$p^2/m \ll mc^2$$

$$\frac{p^2}{2m} \sim \frac{Ze^2}{r} \quad ; \quad \text{but } p \cdot r \sim \hbar \Rightarrow$$

$$\frac{1}{r} \sim \frac{p}{\hbar} \Rightarrow \frac{p^2}{2m} \sim \frac{Ze^2 p}{\hbar} \; ; \; p \sim \frac{mZe^2}{\hbar}$$

$$\frac{m^2 Z^2 e^4}{\hbar^2 m^2 c^2} \ll 1 \Rightarrow \frac{Ze^2}{\hbar c} \ll 1 \Rightarrow Z \ll \frac{1}{\alpha}$$

$\left(\alpha = \frac{e^2}{\hbar c}\right)$. For Z as large as 137 on should solve exact equation. No stable energy levels are found – one particle QM does not work, we get e^+e^- pair creation

$$\varepsilon = m\sqrt{1-(Z\alpha)^2} - m \quad \text{(exact computation)}$$

General relativistic description of bosons and fermions in quantum mechanics.

Bosons

introduce a set of creation and annihilation operators with commutational relations:

$$[a(\vec{k}), a(\vec{k}')] = 0 \, ; \quad [a(\vec{k}), a^+(\vec{k}')] = \delta(\vec{k}-\vec{k}')$$

Construct Hamiltonian

$$\hat{H} = \int d^3k \; \varepsilon_k \, a^+(k) a(k), \quad \text{with } \varepsilon_k = \sqrt{k^2+m^2}$$

and the vacuum state:

$$a(\vec{k})|0\rangle = 0 \quad \text{for all } \vec{k}$$

Eigenstates of H ; $\vec{P}$	Eigenvalues:
$\|0\rangle$	0
$a^+(k)\|0\rangle$	$\varepsilon_k \, , \; \vec{k}$
$a^+(k_1)a^+(k_2)\|0\rangle$	$\varepsilon(k_1)+\varepsilon(k_2), \; \vec{k}_1+\vec{k}_2$
$\ldots$	

Momentum operator:

$$\vec{P} = \int d^3k \; \vec{k} \, a^+(k) a(k)$$

Hilbert space:

$$\Psi = C_0|0\rangle + \int C_1(k)\, a^+(k)|0\rangle d^3k +$$

$$\int C_2(k_1 k_2)\, a^+(k_1) a^+(k_2)|0\rangle d^3k_1 d^3k_2 + .$$

is called Fock state.

- Theory is relativistic
- energy is positive
- probability is positive and is conserved
- Theory is local, it can be shown to coinside with scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2$$

Steps:

Hamiltonian,

φ: coordinate

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} \; : \text{ momentum}$$

Density of Hamiltonian:

$$H = \pi \cdot \dot{\varphi} - \mathcal{L} = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2} m^2 \varphi^2$$

Hamiltonian:

$$\mathcal{H} = \int d^3x \, H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right\} \quad (h)$$

Quantisation:

$$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x}-\vec{y}), \quad (*)$$

commutational relation. Also,

$$[\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = 0$$

Theory in a box of size L:

$$\varphi(x) = \frac{1}{L^{3/2}} \sum_k e^{i\vec{k}\vec{x}} \varphi_{\vec{k}} \quad (1)$$

$$\pi(x) = \frac{1}{L^{3/2}} \sum_k e^{i\vec{k}\vec{x}} \pi_{\vec{k}} \quad (2)$$

$$k_i = \frac{2\pi n_i}{L}, \quad n_i \text{ - integer}$$

$$\varphi_{\vec{k}} = \frac{1}{\sqrt{2\varepsilon_k}} (a_k + a^+_{-k}) \quad (3)$$

$$i\pi_k = \sqrt{\frac{\varepsilon_k}{2}} (a_k - a^+_{-k}) \quad (4)$$

with $[a_k, a^+_{k'}] = \delta_{kk'}$ we reproduce correct commutational relations in $(*)$

$$\varepsilon_k = \sqrt{k^2+m^2}$$

Now, incert (1-4) into h –

we get

$$H = \sum_k \varepsilon_k \left(a_k^+ a_k + \frac{1}{2} \right) -$$

Correct Relativistic Hamiltonian!

Fermions :

We should introduce fermionic creation and annihilation operators.

$\boxed{\text{Fermions}}$: definition: a and a^+ with the following proprietes:
a^+ is hermitean conjugate of a,

$$\{a,a\} = \{a^+,a^+\} = 0 \quad , \quad \text{and } \{a,a^+\} = 1$$

anticommutator:

$$\{a,b\} \equiv ab + ba$$

Consider $N_f = a^+a$

Theorem 1: eigenvalues of N_f are real and non-negative

$$a^+a\,|\alpha\rangle = \alpha|\alpha\rangle \quad :$$

Proof: $\alpha = \langle\alpha|a^+a|\alpha\rangle = \langle a\alpha, a\alpha\rangle \geqslant 0$

Theorem 2: $\alpha = 0$ or 1

Proof: $N_f|\alpha\rangle = \alpha|\alpha\rangle$

$$a^+a\,a|\alpha\rangle = (1-aa^+)a|\alpha\rangle =$$

$$= a|\alpha\rangle - aa^+a|\alpha\rangle =$$

$$= a|\alpha\rangle - a\,\alpha|\alpha\rangle = (1-\alpha)\,a|\alpha\rangle = 0$$

$$N_f\,[a|\alpha\rangle] = (1-\alpha)\,[a|\alpha\rangle] = 0 \qquad (*)$$

also, $a^+a\,a^+|\alpha\rangle = a^+[-a^+a+1]|\alpha\rangle$

$a^+a = 1-aa^+$

$$= a^+(1-\alpha)|\alpha\rangle = a^+|\alpha\rangle \Rightarrow 1-\alpha = 1 \qquad (**)$$

two equations must be satisfied

$(1-\alpha)=0$ or $a|\alpha\rangle=0$ $(*)$

$\alpha=0$ or $a^+|\alpha\rangle=0$ $(**)$

So, our system is two-dimensional –

2 eigenvectors, 2 eigenvalues

if $\alpha=1 \Rightarrow a^+|\alpha\rangle=0$

if $\alpha=0 \Rightarrow a|\alpha\rangle=0 \Leftarrow$ choice to make.

another on i equivalent, after redefinition

Definition: vacuum $|0\rangle$: $a \leftrightarrow a^+$

$a|0\rangle=0$

excited state: $a^+|0\rangle$: $N_f\, a^+|0\rangle = a^+|0\rangle$

Hamiltonian:

$H = \omega a^+a$, $\varepsilon = \omega$ and 0

why this is related to fermions?

or level is empty, or it is occupied once

$(a^+)^2 = 0$!

explicite construction – via 2×2 matrices

$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $a^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $a^+a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $a^+|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Generalisation to many states:

$$\{a_\sigma(\vec{k}), a_\sigma(\vec{k}')\} = 0, \qquad \{a_\sigma(k), a^+_{\sigma'}(k')\} = \delta_{\sigma\sigma'}\delta(k-k')$$

vacuum: $\quad a(k)|0\rangle = 0$

Relativistic description of fermions:

$$\hat{H} = \sum_\sigma \int d^3k\, \varepsilon(k)\left[a^+_\sigma(k)a_\sigma(k) + b^+_\sigma(k)\, b_\sigma(k)\right]$$

$$\{b, b\} = 0, \qquad \{b, b^+\} = \delta(\vec{k}-\vec{k}'),$$

$$\{a, b\} = \{a, b^+\} = 0$$

a: electron

b: positron

One can show that this is equivalent with the action $S = \int d^4x\left[\bar{\psi} i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi\right]$ γ-matrices are related to α and β matrices.

Majorana fermion:

$$H = \int d^3k\, \varepsilon_k \sum_\sigma a^+_\sigma a_\sigma$$

Hilbert space:

$$\psi = c_1|0\rangle + \int c_2\, a^+ d^3k + \int \tilde{c}_1\, b^+ d^3k$$

+ many-body states

This approach allows to describe transitions with change of particle number.

Normalisation:

$$\langle \Psi | \Psi \rangle = 1 \Rightarrow$$

$$|C^0|^2 + \underbrace{\sum_i |C_i^{(1)}|^2}_{\uparrow} + \frac{1}{2!}\sum |C_{ij}|^2 + \dots$$

$\uparrow$ probability to have no particles

probability to have one particle

— " — 2 particles, etc

This allows us to consider processes with changing # of particles:

$$n \to p + e^- + \overline{\nu}_e \; ;$$

$$H_{int} = a_n^+ a_p a_e a_{\bar{\nu}}$$

Higgs decay, $H \to e^+ e^-$

$$H_{int} = a_H^+ a_{e^-} a_{e^+} + a_{e^-}^+ a_{e^+}^+ a_H$$

etc.