Quelques exercices supplémentaires

Exercice 15.1. Let M be an oriented manifold with boundary. Show that M has a natural orientation.

Exercice 15.2. Show that for k-form α and $n-k-1$ form β

$$\int_M \alpha \wedge d\beta = (-1)^k \left[\int_M \alpha \wedge \beta - \int_M d\alpha \wedge \beta \right].$$

This is the *integration by parts* formula for differential forms.

Exercice 15.3. Consider the form $\omega \in \Omega^{n-1}(^n \backslash \{0\})$

$$\omega = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^i}{r^n} dx^{N \backslash \{i\}},$$

where

$$r = \sqrt{\sum_i (x^i)^2},$$

and $N = \{1, \ldots, n\}$. Show that $d\omega = 0$ Show using Stokes' theorem that $H^{n-1}(^n \backslash \{0\})$ is non-zero.

Hint. First recall that if $\omega = d\xi$ then $i^*(\omega) = i^*(d\xi) = di^*(d\xi)$, where $i : S^{n-1} \to^n \backslash \{0\}$ is the inclusion map. In words: if the form is exact then it's restriction is exact. Now let

$$\tilde{\omega} = \sum_i (-1)^{i+1} x^i dx^{N \backslash \{i\}}.$$

Then $i^*(\omega) = i^*(\tilde{\omega})$ because $r \circ i \equiv 1$. Apply Stokes' theorem to $\tilde{\omega}$.

Exercice 15.4. Recall Green's theorem

$$\iint_\Omega \left(\frac{\partial F^1}{\partial x^2} - \frac{\partial F^2}{\partial x^1} \right) dx^1\, dx^2 = \oint_{\partial\Omega} F \cdot dl$$

Gauss' theorem

$$\iiint_\Omega \nabla \cdot F\, dx^1\, dx^2\, dx^3 = \oiint_{\partial\Omega} F \cdot \nu dS$$

and Stoke's theorem (from vector calculus)

$$\iint_\Sigma \nabla \times F \cdot \nu dS = \oint_\Sigma F \cdot dl.$$

See *Analyse avancée pour ingénieurs* Chap. 4,6,7. Identify how each of these is Stokes' theorem for manifolds by identifying F with a differential form, what are the manifolds of integration and what are the boundaries. Calculate dF in each case.