Quelques exercices supplémentaires

Exercice 15.1. Let M be an oriented manifold with boundary. Show that M has a natural orientation.

Solution 15.1. Let $F : U \subset \overline{\mathbb{H}}^n \to V \subset \overline{\mathbb{H}}^n$ be a diffeomorphism. which sends $U \cap \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^{n-1} \times \{0\}$. Then let $F_\partial := F|_{\mathbb{R}^{n-1} \times 0}$. It follows that $\partial_j|_q F^n = 0$ for any $q \in \mathbb{R}^{n-1} \times \{0\}$. So it follows that $\det DF(q) = (\partial_n|_q F^n) \det DF_\partial(q)$. Finally ∂_n cannot be negative, as then $F(U)$ would not be contained in the upper half plane. Now let $\varphi : U \to \overline{\mathbb{H}}^n$ and $\psi : V \to \overline{\mathbb{H}}^n$, and let $\varphi_\partial = \varphi|_{\partial M \cap U}$. Let $\Phi_{\partial UV}$ be the restriction of Φ_{UV} to $\varphi(\partial M \cap U \cap V)$. Then by the previouse

$$0 < \det D\Phi_{UV}(\varphi_V(q)) = \partial_n|_{\varphi_V(q)} \Phi^n_{UV} \det D\Phi_{\partial UV}(\varphi_V(q).$$

But as before $\partial_n \Phi^n_{UV} \geq 0$, and so $\partial_n \Phi^n_{UV}$ and $\det D\Phi_{\partial UV}$ are strictly positive. Consequently ∂M has a transition maps with positive Jacobian determinant ($\det D\Phi_{\partial UV}$).

Exercice 15.2. Show that for k-form α and $n-k-1$ form β

$$\int_M \alpha \wedge d\beta = (-1)^k \left[\int_M \alpha \wedge \beta - \int_M d\alpha \wedge \beta \right].$$

This is the *integration by parts* formula for differential forms.

Solution 15.2. Recall that $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{k(n-k-1)} \alpha \wedge d\beta$. If we apply Stokes' theorem to both sides, we get

$$\int_M \alpha \wedge \beta = \int_M d\alpha \wedge \beta + (-1)^k \int_M \alpha \wedge d\beta,$$

which yields the result after rearranging and multiplying by $(-1)^k$

Exercice 15.3. Consider the form $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$

$$\omega = \sum_{i=1}^n (-1)^{i+1} \frac{x^i}{r^n} dx^{N \setminus \{i\}},$$

where

$$r = \sqrt{\sum_i (x^i)^2},$$

and $N = \{1, \ldots, n\}$. Show that $d\omega = 0$ Show using Stokes' theorem that $H^{n-1}(\mathbb{R}^n \setminus \{0\})$ is non-zero.

Hint. First recall that if $\omega = d\xi$ then $i^*(\omega) = i^*(d\xi) = di^*(d\xi)$, where $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the inclusion map. In words: if the form is exact then it's restriction is exact. Now let

$$\tilde{\omega} = \sum_i (-1)^{i+1} x^i dx^{N \setminus \{i\}}.$$

Then $i^*(\omega) = i^*(\tilde{\omega})$ because $r \circ i \equiv 1$. Apply Stokes' theorem to $\tilde{\omega}$.

Solution 15.3. First

$$\begin{aligned}
d\omega &= \sum_i (-1)^{i+1}\frac{1}{r^n}dx^i \wedge dx^{N\setminus\{i\}} - (-1)^{i+1}(n)\frac{x^i}{r^{n+2}}dr \wedge dx^{N\setminus\{i\}} \\
&= \frac{n}{r^n}dx^N - n\sum_{i=1}^{n}(-1)^{n+1}\frac{(x^i)^2}{r^{n+2}}dx^i \wedge dx^{N\setminus\{i\}} \\
&= (\frac{n}{r^n} - \frac{nr^2}{r^{n+2}})dx^N = 0.
\end{aligned}$$

Now consider the the restriction of ω to S^{n-1}. If we can show the integral is non-zero, then we are done. But for $r = 1$ this is the integral of the form $\Omega = \sum_i (-1)^{i+1}x^i dx^{N\setminus\{i\}}$:

$$\begin{aligned}
\int_{S^{n-1}} \omega &= \int_{S^{n-1}} \Omega \\
&= \int_{B^n} \Omega \\
&= \int_{B^n} d\Omega.
\end{aligned}$$

But $d\Omega = ndx^N$, so $\int_{S^{n-1}} \omega = n\mathrm{Vol}(B^n) \neq 0$. And so ω cannot be exact.

Exercice 15.4. Recall Green's theorem

$$\iint_\Omega \left(\frac{\partial F^1}{\partial x^2} - \frac{\partial F^2}{\partial x^1}\right)\, dx^1\, dx^2 = \oint_{\partial\Omega} F \cdot dl$$

Gauss' theorem

$$\iiint_\Omega \nabla \cdot F\, dx^1\, dx^2\, dx^3 = \oiint_{\partial\Omega} F \cdot \nu dS$$

and Stoke's theorem (from vector calculus)

$$\iint_\Sigma \nabla \times F \cdot \nu dS = \oint_\Sigma F \cdot dl.$$

See *Analyse avancée pour ingénieurs* Chap. 4,6,7. Identify how each of these is Stokes' theorem for manifolds by identifying F with a differential form, what are the manifolds of integration and what are the boundaries. Calculate dF in each case.

Solution 15.4. For Green's theorem, Ω is a two dimensional manifold and F is a one form

$$F = F^1 dx^1 + F^2 dx^2.$$

Then

$$dF = \left(-\frac{\partial F^1}{\partial x^2} + \frac{\partial F^2}{\partial x^1}\right) dx^1 \wedge dx^2.$$

And so this is

$$\int_\Omega dF = \int_{\partial\Omega} F$$

for F a one-form and Ω a 2-manifold. For Gauss' theorem

$$F = F^1 dx^2 \wedge dx^3 - F^2 dx^1 \wedge dx^3 + F^3 dx^1 \wedge dx^2.$$

From Exercise 12.5, we recall that $dF = \nabla \cdot F$, and so Gauss' theorem corresponds to

$$\int_{\Omega} dF = \int_{\partial\Omega} F,$$

for F a 2 form and Ω a 3-manifold.
For Stokes' theorem let

$$F = F^1 dx^1 + F^2 dx^2 + F^3 dx^3,$$

then dF corresponds to $\nabla \times F$, and it corresponds to Stokes' theorem on a 2-manifold for a one form F.