QUANTUM PHYSICS III

Appendix B 13 October 2017

We want to justify the approximation for the slit we made when evaluating the transition amplitude in Ex.3. To this end, one should construct a family of orthogonal functions such that the original slit is represented by a convergent series of these functions with the first term being approximately the Gaussian function $\tilde{G}(y) = \exp(-y^2/2b^2)$. For simplicity, we change the variable to $x = y/b$. Then

$$G(x) = \sum_{i=0}^{\infty} c_i f_i(x) \ , \quad c_i = (f_i, G) \ , \tag{1}$$

where c_i are found by taking a suitable scalar product of the expression above, with respect to which the functions f_i are orthonormal. One wants

$$c_0 f_0 \approx \tilde{G}(x) \ . \tag{2}$$

Since $G(x)$ is even, it is enough to build a family of functions which is dense in the space of analytical even functions. We take the probe monomial functions of the form

$$e^{-x^2/2} \ , \quad x^2 e^{-x^2/2} \ , \quad x^4 e^{-x^2/2} \ , \ldots \tag{3}$$

and apply the Gram-Schmidt orthogonalization procedure with the scalar product

$$(f, g) = \int_0^{\infty} f(x) g(x) dx \ . \tag{4}$$

The first few resulting polynomials and the corresponding coefficients in eq. (1) are

$$\begin{aligned}
f_0 &= \frac{\sqrt{2}}{\pi^{1/4}} e^{-x^2/2} \ , & c_0 &\approx 0.91 \ , \\
f_1 &= \frac{1}{\pi^{1/4}} e^{-x^2/2}(-1 + 2x^2) \ , & c_1 &\approx -0.27 \ , \\
f_2 &= \frac{1}{2\sqrt{3}\pi^{1/4}} e^{-x^2/2}(3 - 12x^2 + 4x^4) \ , & c_2 &\approx 0.03 \ , \\
&\ldots & &\ldots
\end{aligned} \tag{5}$$

Shown on figure below are several partial sums $G_n(x) = \sum_{i=0}^{n} c_i f_i(x)$. We observe that they do experience a convergence to the original step function, although not uniform, since the slit does not belong to the space of functions in which the set (5) is complete. When calculating the transition amplitude in Ex.3, the terms $c_i f_i$ with $i > 0$ in the expansion (1) get suppressed by the small constants c_i, and, moreover, each monomial of the form $x^{2n} e^{-x^2/2}$ yields additional suppression by a factor of $\hbar^{2n}$.

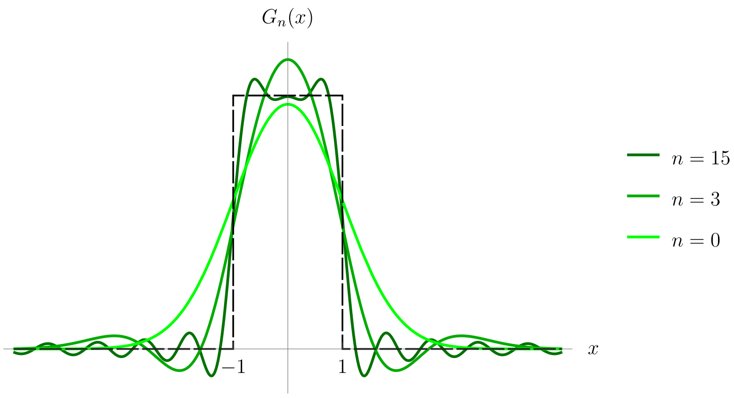

FIGURE 1 – The partial sums made of analytic functions, that approximate the diffraction slit drawn by a dashed line.