Networks out of Control:

Homework Set 4

Exercise 1

Suppose G is an undirected graph with n nodes such that each node of G has a local clustering coefficient of zero.

1. Prove that the number of edges in G is at most $\frac{n^2}{4}$.
2. For any even integer $n \geq 4$, construct such a graph.

Note that this implies that dense graphs may have small clustering coefficients!

Hint: you may find useful the following inequality for the geometric and arithmetic mean of two non-negative numbers x, y: $\sqrt{xy} \leq (x+y)/2$.

Exercise 2

Consider a network composed of individuals of two types, A and B. Let a be the number nodes of type A and let b be the number nodes of type B, with $a \geq b \geq 4$ without loss of generality. Given node v, let d_v be the degree of v, and let s_v be the degree of v when restricted to the network consisting only of nodes of the same type as v. Assume that $\sum_{v \in \mathcal{I}} d_v > 0$ for $\mathcal{I} \in \{A, B\}$; i.e., in each group there exists at least one vertex with nonzero degree. Then, we can define a *homophily index* of group $\mathcal{I} \in \{A, B\}$ as follows:

$$h_{\mathcal{I}} = \frac{\sum_{v \in \mathcal{I}} s_v}{\sum_{v \in \mathcal{I}} d_v}.$$

Clearly $0 \leq h_{\mathcal{I}} \leq 1$.

Answer the following questions:

1. What is the minimum and maximum global clustering coefficient for a network with $h_A = h_B = 1$?
2. What is the minimum and maximum global clustering coefficient for a network with $h_A = h_B = 0$?
3. Assume that friendships are formed uniformly at random irrespective of type. What are h_A and h_B?
4. Recall the homophily threshold value $2pq$ as seen in class where p is the fraction of type A nodes, q is the fraction of type B nodes in the network, and $2pq$ is the fraction of edges we expect to see in a network that exhibits no homophily or heterophily.

- Prove that if $a = b$ and h_A and h_B are as found in part 3 above, then the network exhibits no homophily nor heterophily (i.e., exactly a $2pq$ fraction of edges are across types),
- Show that the contrapositive is not true; i.e., there are networks with $a = b$ that exhibit no homophily nor heterophily but does not have h_A or h_B as found in part 3 above.

Exercise 3

Recall the network considered in class in which nodes are laid out on a square grid, each node is connected to all neighbors within some distance r in the ℓ_∞ metric, and, in addition, for each node we add k edges whose endpoint is distributed according to

$$\frac{d(u,v)^{-\gamma}}{\sum_{v \neq u} d(u,v)^{-\gamma}},$$

Show that, for $\gamma < 2$, decentralized routing algorithms are inefficient. In particular, the expected delivery time of any decentralized algorithm for $0 \leq \gamma < 2$ is $\Omega(n^\delta)$, for $\delta = (2-\gamma)/3$.

Hint: We showed this for $\gamma = 0$ in class (which corresponds to the Watts-Strogatz model before we imposed a distance metric); the proof here should follow the same approach.