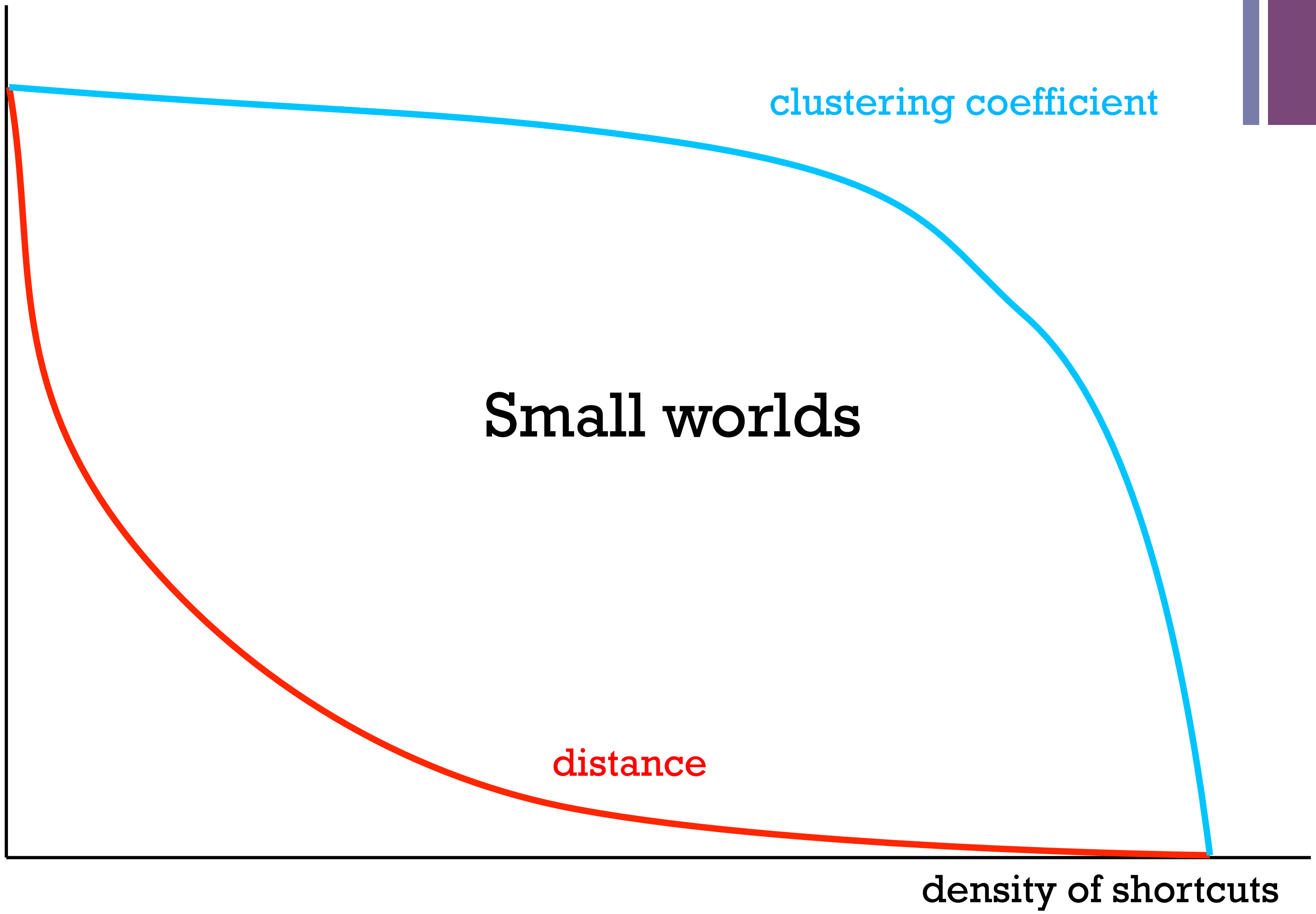
Networks Out of Control: Real-World Networks 3

Real-World Example 2: Social Networks

Social Network

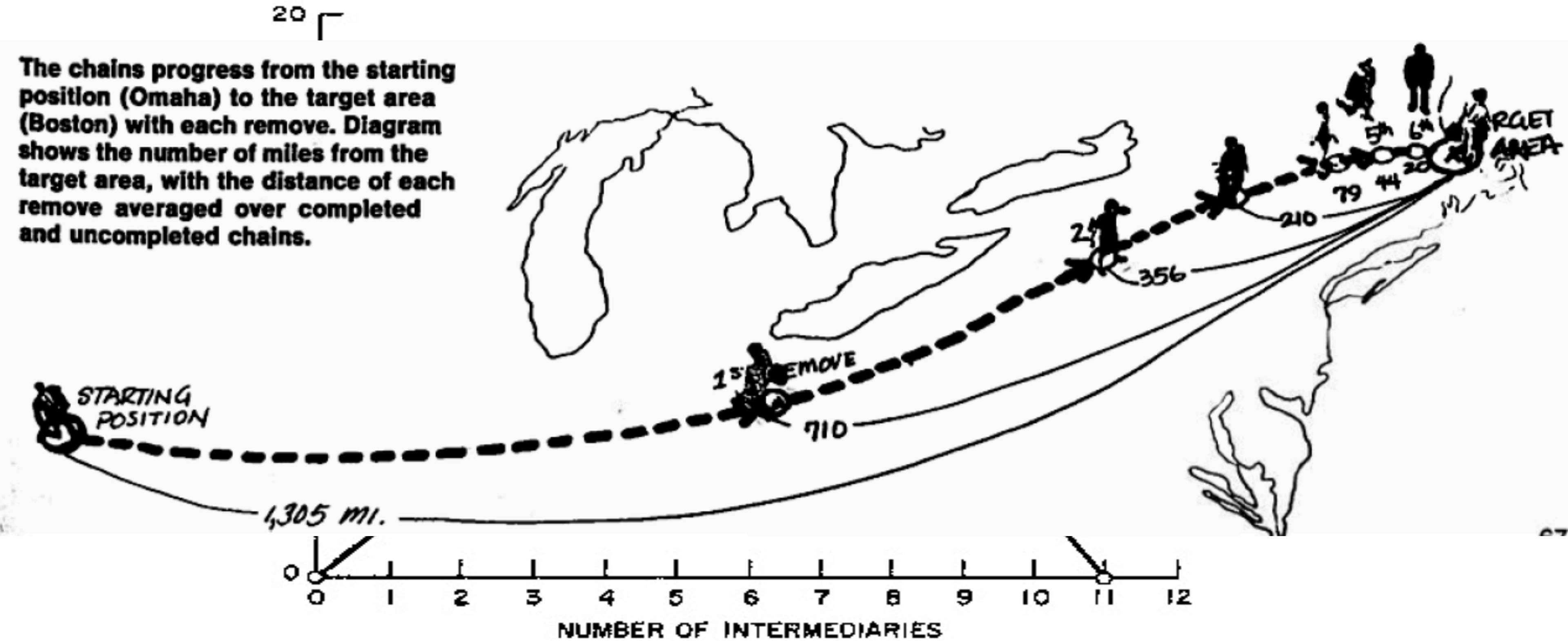
- Nodes are people. (Undirected) edges are connections representing friendships, acquaintances, business relationships, etc.
- Properties:
  - Size
  - Connected Structure
  - Degree Distribution
  - (Small) Diameter
  - Clustering
  - Navigability
  - Homophily
  - Betweenness, Strong/Weak Ties, Power Imbalance, Partitioning, etc…

Clustering and Distance in SW Network



Social Networks: Small-World

- [Milgram 1969] experiment to study the *average distance* between two nodes in a social network.



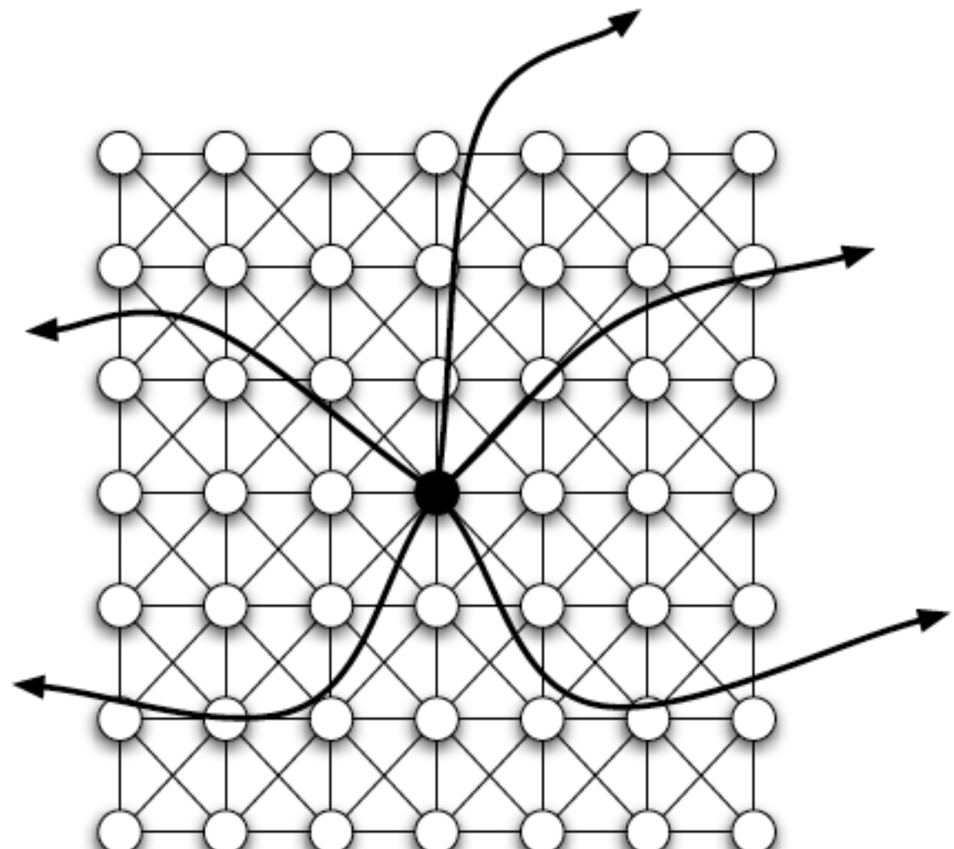
- Paths are not just short – they can be found!

Social Networks: Small-World

- A *decentralized routing algorithm* takes local (node-level) decisions on where to forward a message next based only on
  - the geographic location of the current node
  - the geographic locations of its neighbors,
  - the geographic location of the target node
- What do we mean by ``geographic location”?

Watts-Strogatz on a Grid

- There are n^2 nodes arranged in a square grid in R^2, and we endow the space with the l_1 norm.
- Every node v connects to all nodes u such that $d(u,v) \leq r$.
- Every node has k additional edges connected to uniformly random endpoints u.



Is Watts-Strogatz on a Grid Navigable?

- Note: the distance between a (randomly selected) source and target is O(n) a.a.s.
- Our goal: Reach the target in $O(n^{\delta})$ steps for $\delta << 1$.
- Our approach:
  - Consider a ball B of nodes within some "short" distance n^{δ} to the target.
    - Within the ball, can reach the target in n^{δ} steps.
  - Can we reach the ball quickly?
    - Without shortcuts takes $O(n - n^{\delta})$ steps to reach the ball a.a.s.
    - Must make use of shortcuts, in particular, need to show that at least one of the first $O(n^{\delta})$ nodes has a shortcut to B.

Is Watts-Strogatz on a Grid Navigable?

- How many vertices are there in B? $1 + \sum_{j=1}^{n^{\delta}} 4j \leq 2n^{2\delta}$
- What is the probability that a vertex v has a shortcut into B?

$$\mathbb{P}[E_v] = k\frac{|B|}{n^2} \leq 2kn^{2\delta-2}$$

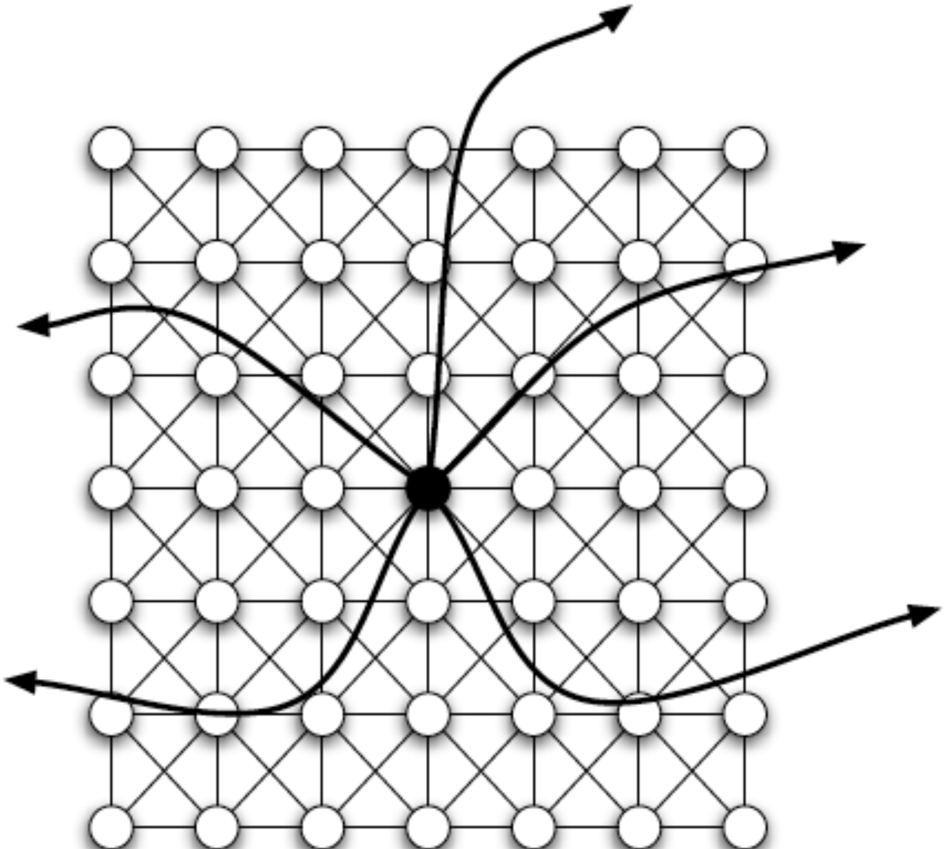
- What is the probability that any vertex in the first t ($= \lambda n^{\delta}$) steps has a shortcut into B?

$$\mathbb{P}[E = \cup_{1 \leq i \leq t} E_{v_i}]$$

- If $3\delta - 2 < 0$ (i.e., $\delta < ⅔$) this probability is vanishing.
- Thus, routing takes (at least!) $n^{⅔}$ steps!

Distance-Proportional Watts-Strogatz on a Grid

- There are n^2 nodes arranged in a square grid in R^2, and we endow the space with the l_1 norm.
- Every node v connects to all nodes u such that $d(u,v) \leq r$.
- Every node has k additional edges connected i.i.d. to u **proportionally to $d(u,v)^{-\gamma}$ for constant $\gamma \geq 0$.**



$$\frac{d(u,v)^{-\gamma}}{\sum_{u \neq v} d(u,v)^{-\gamma}}$$

Is Distance-Proportional Watts-Strogatz Navigable?

- When γ =0
  - The model is exactly the original WS model on the grid, so still not navigable!
  - In fact, a similar proof shows it is not navigable for $\gamma < 2$; the number of steps is at least n^{δ} for $\delta = (2 - \gamma)/3$.
- What is the problem?
  - Shortcuts are ``too random''
- What about $\gamma > 2$?

Is Distance-Proportional Watts-Strogatz Navigable?

$$\mathbb{P}\left\{E_v^d\right\} \leq$$

- Let E_{vi} denote the event that, at step i the vertex v_i has a shortcut of length at least $n^{1-\beta}$, and E the event that this occurs in the first λn^{β} steps.

$$\mathbb{P}[E] \leq \sum_{i=1}^{\lambda n^{\beta}} \mathbb{P}[E^d_{v_i}]$$

- For this to not be vanishing, β must be at least $(\gamma-2)/(\gamma-1)$.
- Thus, the message can at best find shortcuts of distance less than $n^{1-\beta}$ in the first λn^{β} steps, for a total progress O(n), so at least $O(n^{\beta})$ steps are required.

Is Distance-Proportional Watts-Strogatz Navigable?

- When $\gamma < 2$, shortcuts are ``too random''
- When $\gamma > 2$, shortcuts are ``too short''
- Is there a sweet-spot at $\gamma = 2$?

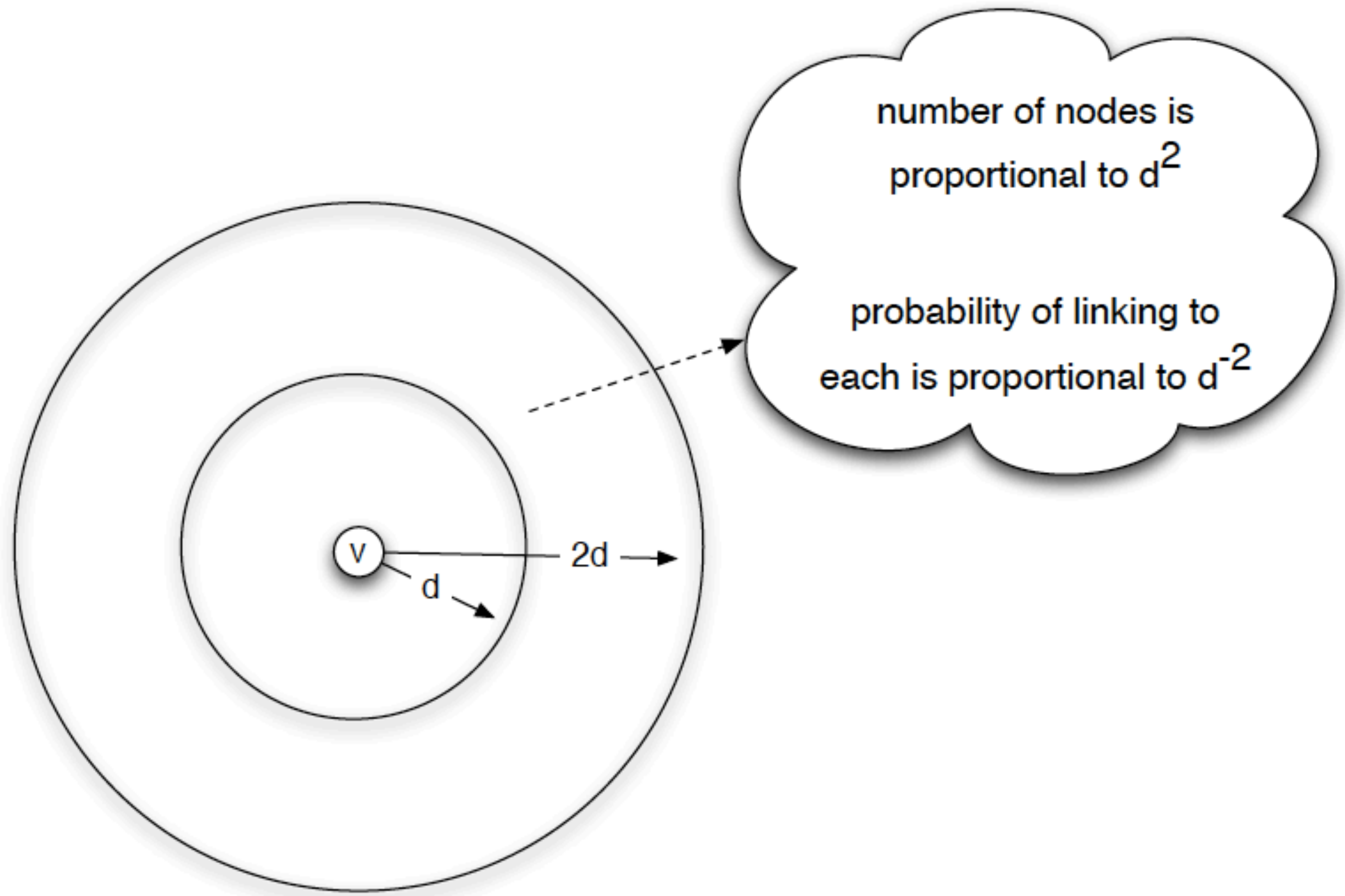
Is Distance-Proportional Watts-Strogatz Navigable?

- Show constructive proof for $r=1$ and $k=1$ (this is the worst case) with steps $O((\log n)^2)$.
- At each time step, send to the neighbor that is closest to the target.
  - Note: this always terminates as progress is made at every step, if only through the lattice.
- Definitions:
  - The annuli U_j is the set of nodes at lattice distance in $[2^j+1,\ 2^{j+1}]$ from the target.
  - The ball B_i is the union of all U_j with $j < i$.
  - The algorithm is in phase j when the message is in U_j
  - Note: there are at most log n phases.

Is Distance-Proportional Watts-Strogatz Navigable?

- Approach: Show that we progress quickly from phase to phase.
- If we are in phase j at node u, then to end the phase, we must pass the message to a node in B_j.
  - The probability of connecting to B_j is at least: $|B_j| \cdot \frac{\max_{v \in B_j} d(u,v)^{-\gamma}}{\sum_{u \neq v} d(u,v)^{-\gamma}}$
  - The size of B_j: $|B_j| \geq 2^{2j-1}$
  - The maximum distance: $\max_{v \in B_j} d(u,v)^{-\gamma} \leq 2^{j+1} + 2^j < 2^{j+2}$
  - The normalizing constant is:

Is Distance-Proportional Watts-Strogatz Navigable?



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  - The normalizing constant is: $\sum_{u \neq v} d(u,v)^{-\gamma} \leq 4 \ln(6n)$

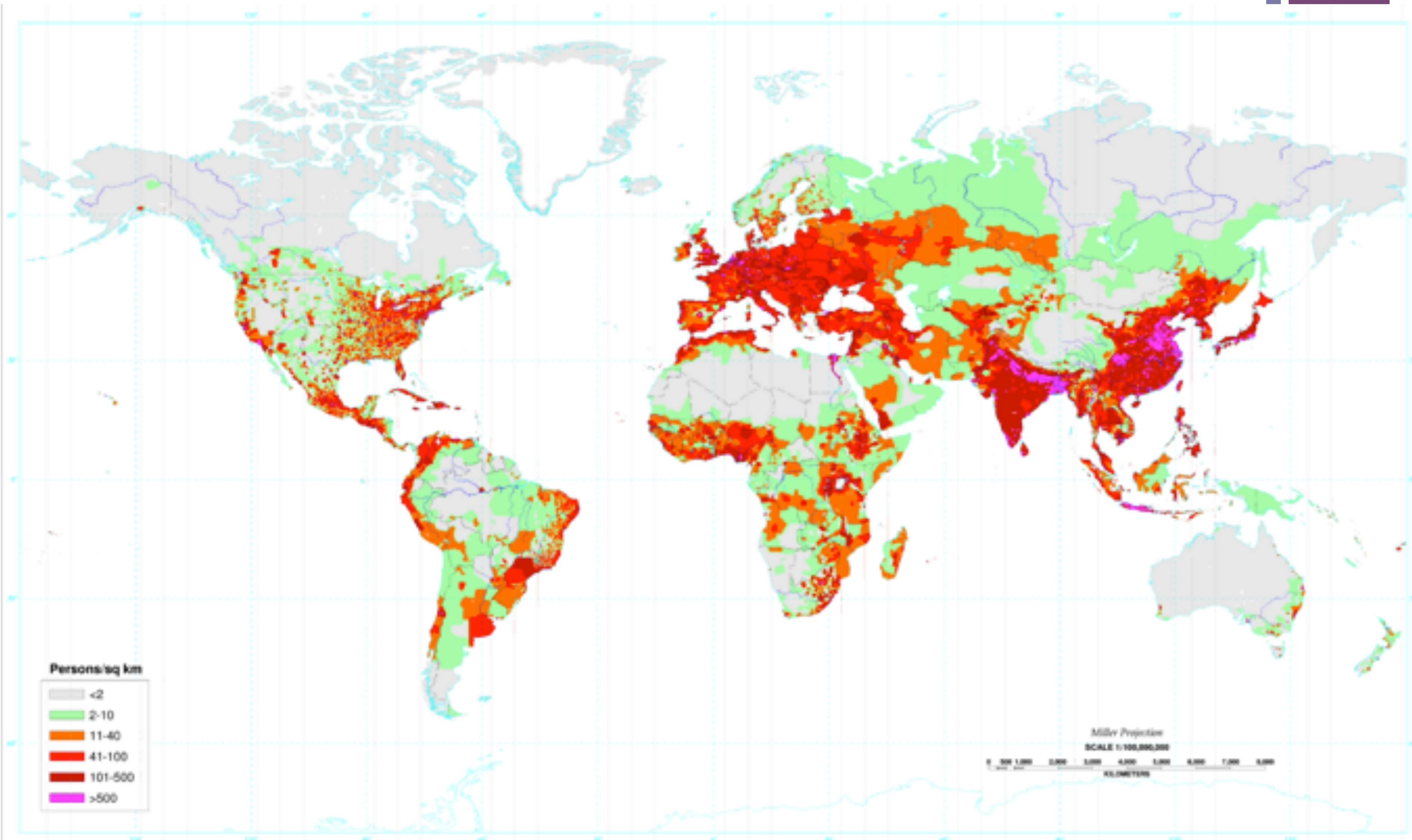
Is Distance-Proportional Watts-Strogatz Navigable?

- Therefore, we change phases with probability on the order of 1/log(n).
- In expectation, it takes O(log(n)) steps to change phases, and as we noted before, there are log(n) phases, hence we have an efficient decentralized routing algorithm that requires $O((\log n)^2)$ steps!

Is Distance-Proportional Watts-Strogatz Navigable?

- Navigability:
  - When $\gamma < 2$, shortcuts are ``too random''
  - When $\gamma > 2$, shortcuts are ``too short''
  - There a sweet-spot at $\gamma = 2$!
- In general, can have a d-dimensional lattice, and have a similar phase transition at $\gamma = d$.
- Can also take other underlying topologies (e.g., see a case for trees in the notes).

Are real-world networks Navigable?



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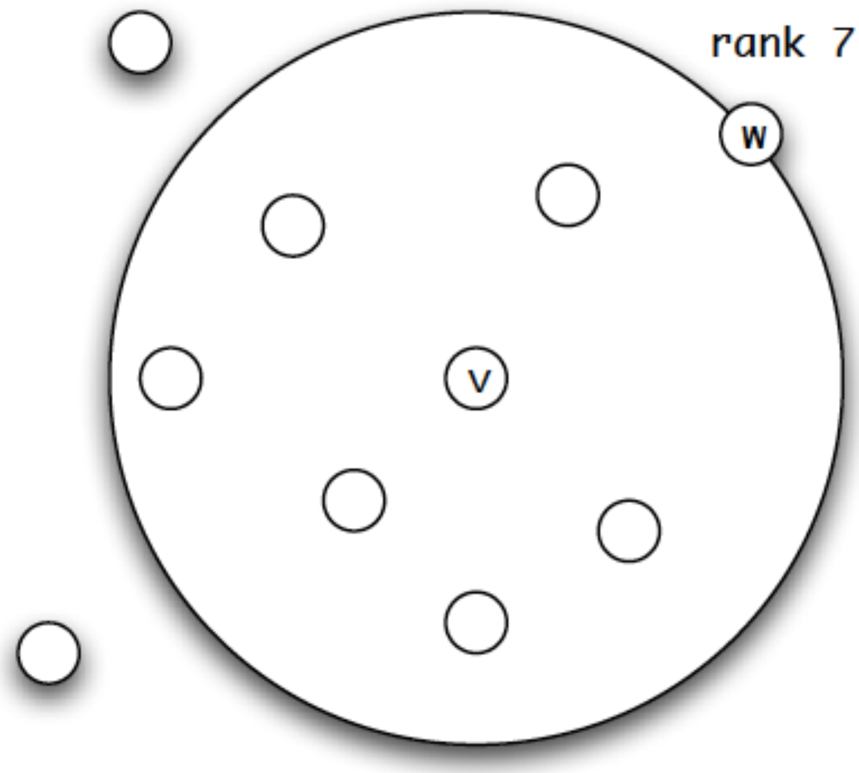
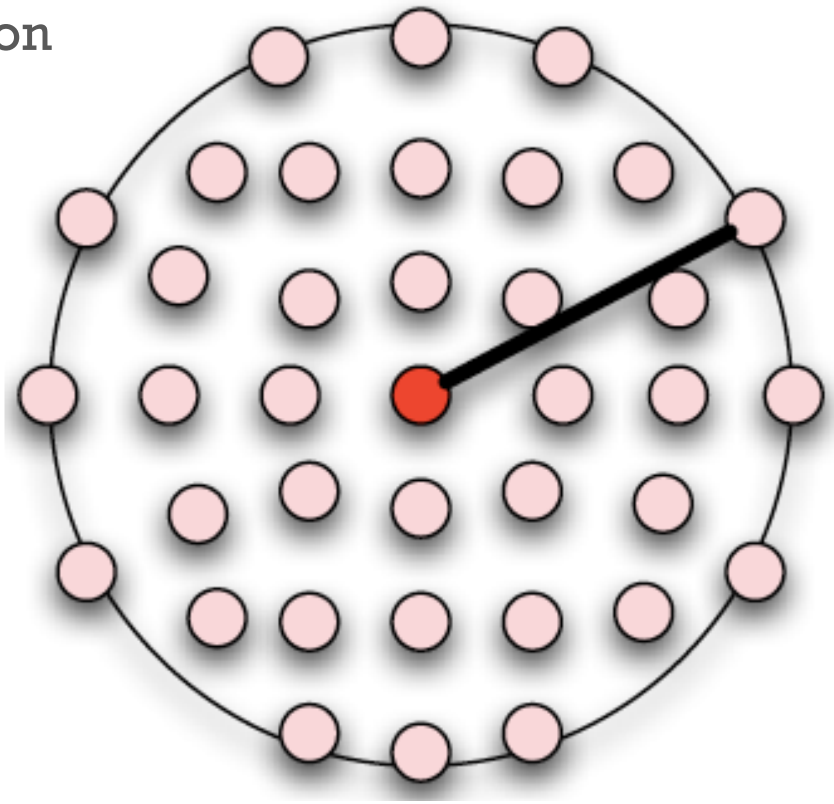


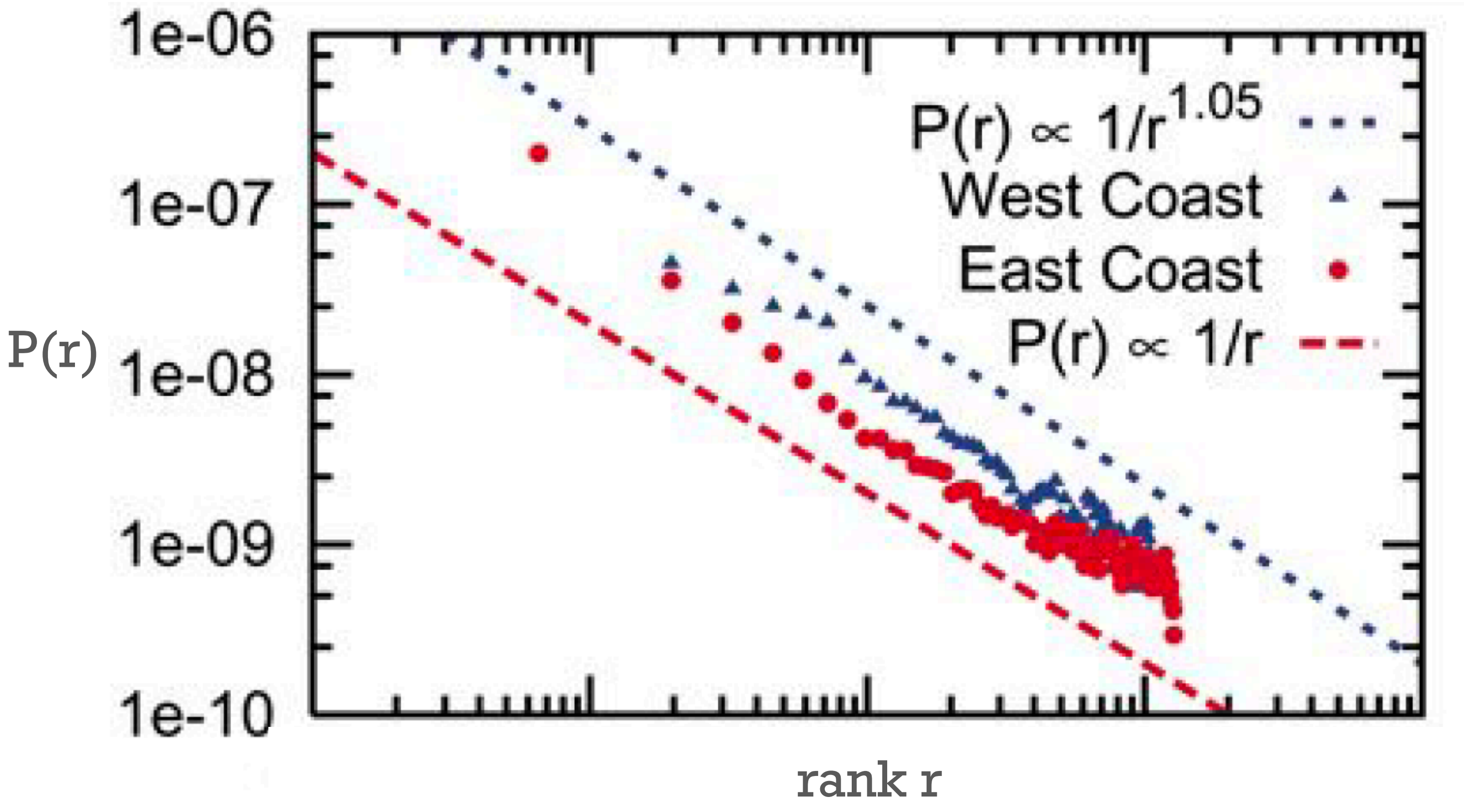
Are real-world networks Navigable?

- Phase transition at $\gamma = 1$ with respect to the rank!



Are real-world networks Navigable?

- LiveJournal data:



Are real-world networks Navigable?

- Does this mean we can reach any target in a social network via decentralized search?
- Attempts to replicate Milgram's experiment have had mixed results.
- In particular, completion rates vary dramatically:
  - Highest for individuals with high social visibility, e.g., professors and journalists.
- In our models, the networks were (effectively) symmetric – this need not be the case in general!