Networks Out Of Control: Evolution & Dynamics 2
Network Formation Games

Network Formation

- **Models for Networks:**
  - G(n,p)
  - G(n,r)
  - Watts-Strogatz
  - Preferential Attachment
- **Commonality:**
  - Defined by random processes.
- **Alternative mentality:**
  - Consider *motivations* of participants in the network, and model behavior as a function of these goals.
  - Will use game theory!

Games

- **Agents:** two or more participants.
- **Strategies:** options available to the agent.
- **Outcome:** global end result (function of all agent's actions).
- **Utility:** real-valued function of outcome for a given agent.
- In game theory:
  - Agents want to maximize their own utility (selfish behavior)
  - We (often) study optimality, i.e., outcomes such that the average utility is maximized.
  - We (often) study equilibira, i.e., outcomes such that no agent benefits by changing their action.

Bimatrix Game

Example 1: prisoner's dilemma

- **Agents:** two friends commit a crime and are caught.
- **Actions:** confess to the police or maintain innocence.
- **Payoffs:**

	Confess	**Lie**
Confess	-5, -5	-1, -10
Lie	-10, -1	-2, -2

- **Equilibrium:** no one wants to *defect* (i.e., change their strategy).

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Bimatrix Game

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- **Equilibrium:** no one wants to *defect* (i.e., change their strategy).
- In fact, this game has a **Dominant Strategy Equilibrium**.

Bimatrix Game

Example 2: coordination game

- **Agents:** you and your best friend are going for lunch.
- **Actions:** go to the Thai Food truck or go to the Pizza truck.
- **Payoffs:**

	Thai Food	Pizza
Thai Food	2, 3	1, 1
Pizza	-1, -1	3, 2

- **Equilibrium:** no one wants to *defect*.

Bimatrix Game

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- **Actions:** go to the Thai Food truck or go to the Pizza truck.
- **Payoffs:**

	Thai Food	Pizza
Thai Food	2, 3	1, 1
Pizza	-1, -1	3, 2

- **Equilibrium:** no one wants to *defect*.
  - No Dominant strategy equilibrium.
  - Two pure Nash equilibrium.

Bimatrix Game

Example 3: zero-sum game

- **Agents:** you and your worst enemy are going for lunch.
- **Actions:** go to the Thai Food truck or go to the Pizza truck.
- **Payoffs:**

	Thai Food	**Pizza**
Thai Food	-1, 1	1, -1
Pizza	1, -1	-1, 1

- **Equilibrium:** no one wants to *defect.*

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Bimatrix Game

Example 3: zero-sum game

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- **Actions:** go to the Thai Food truck or go to the Pizza truck.
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	Thai Food	Pizza
Thai Food	-1, 1	1, -1
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- **Equilibrium:** no one wants to *defect.*
  - no *pure* Nash equilibrium.

Bimatrix Game

Example 3: zero-sum game

- **Agents:** you and your worst enemy are going for lunch.
- **Actions:** go to the Thai Food truck or go to the Pizza truck.
- **Payoffs:**

	Thai Food	Pizza
Thai Food	-1, 1	1, -1
Pizza	1, -1	-1, 1

- **Equilibrium:** no one wants to *defect.*
  - no *pure* Nash equilibrium.
  - However, *mixed* Nash equilibrium (each chosen with probability ½).

Game Theory

- Any interaction can be modeled by a game
  - In particular: Networks!

Network Formation Games

- **Agents:** $N = \{1, \ldots, n\}$
- **Strategies:** A subset S_i of N x N (potential edges)
  - We also refer to the strategy vector $S = (S_1, S_2, \ldots, S_N)$.
- **Outcome:** A network G where V = N and $E = \cup_i S_i$.
- **Utility:** real-valued functions u_i of the network G .
- Objectives:
  - Agents select S_i in order to maximize u_i (selfish behavior)
  - We study optima: graphs G such that the sum of utilities is maximized.
  - We study stability: graphs G such that no agent wants to change their S_i.

Network Formation Games

- Local connection game (social / information networks)
  - Agent can build edges from itself to other nodes (at a cost), and wants to be connected to all nodes via short paths.
- Global connection game (infrastructure networks)
  - Agents are no longer nodes, each agent wants to ensure some s-t path is built, and can build edges anywhere (at a shared cost).

Local Connection Game

Local Connection Game

- Local connection game (social networks)
  - Agent can build edges from itself to other nodes (at a cost), and wants to be connected to all nodes via short paths.

- Each of the n agents is a node.
  - The strategy space for node *u* is a subset S_u of V.
    - This corresponds to building uv edges for all v in S_u.
  - Each edge has a cost α
  - The total distance to other nodes (using all edges) also incurs a cost.
  - Overall, node *u* wishes to minimize:
    $$\alpha n_u + \Sigma_v d(u,v)$$
    where n_u is the size of S_u and $d(u,v)$ is the distance between u and v.

- The cost of the network is the sum of the costs of all agents:
  $$\Sigma_u(\alpha n_u + \Sigma_v d(u,v))$$
  $$= \alpha m + \Sigma_{u,v} d(u,v)$$

where $m = |E|$.

Optimal Networks

- An optimal network minimizes:

$$\alpha m + \Sigma_{u \neq v} d(u,v)$$

- Goal: Characterize optimal networks.
- Lemma: optimal networks
  - If $\alpha \leq 2$, then the complete graph is an optimal network.
  - If $\alpha \geq 2$, then the star network is an optimal network.
- Approach:
  - Lower bound the cost,
  - Give network(s) that attain the lower bound.

Stable Networks

- Recall: Node *u* wishes to minimize:

$$\alpha n_u + \Sigma_v d(u,v)$$

- Goal: Find some stable networks
- Approach:
  - Is the complete graph stable?
  - Is the star graph stable?
    - Assume center node pays for all edges.
    - (in fact true for any star)
- Lemma:
  - If $\alpha \leq 1$, then the complete graph is stable.
  - If $\alpha \geq 1$, then the star graph is stable.

Price of Stability

- The Price of Stability is the ratio: $\frac{\text{value of best equilibrium}}{\text{value of optimal solution}}$
- Lemma:
  - If $\alpha \geq 2$, then the optimal network is a star.
  - If $\alpha < 2$, then the optimal network is a complete graph.
- Lemma:
  - If $\alpha \geq 1$, then any star is a Nash equilibrium.
  - If $\alpha \leq 1$, then the complete graph is a Nash equilibrium.
- Theorem:
  - For $\alpha \geq 2$ and $\alpha \leq 1$, the price of stability is 1.
  - For $1 < \alpha < 2$?
    - Recall: opt value $\geq (\alpha-2)m + 2n(n-1)$
    - The price of stability is at most 4/3.

Exercise

Local Connection Game

- Show that for $\alpha < 1$ the complete graph is the unique equilibrium.
  - Does this imply anything about the price of anarchy?

- Construct a Nash equilibrium that is not a star for *some* N and *some* $\alpha > 2$.

Global Connection Game

Global Connection Game

- Agents are no longer nodes in the network, external agents with some desire over global properties of the network.
  - For example, the vertices are neighborhoods and edges are roads, and you would like the path from your home to your office to be well-maintained.

- There are k agents, each with a source s_i and sink t_i node.
  - The strategy space for agent *i* is the set of all paths P from s_i to t_i .
    - We let P_i be the path she selects.
  - Let k_e be the number of agents using edge e:

$$u_i = \Sigma_{e \text{ in } P_i} c_e/k_e$$

- The sum of the agents costs is

$$\Sigma u_i = \Sigma_{e \text{ in some } P_i} c_e .$$

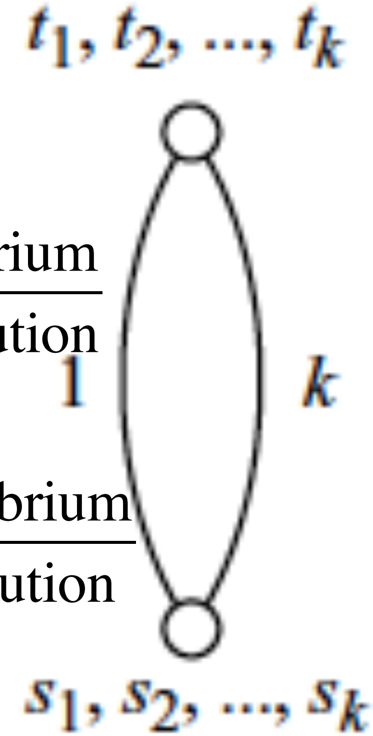
  - Maximizing this quantity is known as the Steiner tree problem.

Price of Anarchy

- For the given example:
  - What is the optimal solution?
  - What are the equilibria?
  - What is the price of stability?
  - What is the price of anarchy?

$$\frac{\text{value of best equilibrium}}{\text{value of optimal solution}}$$

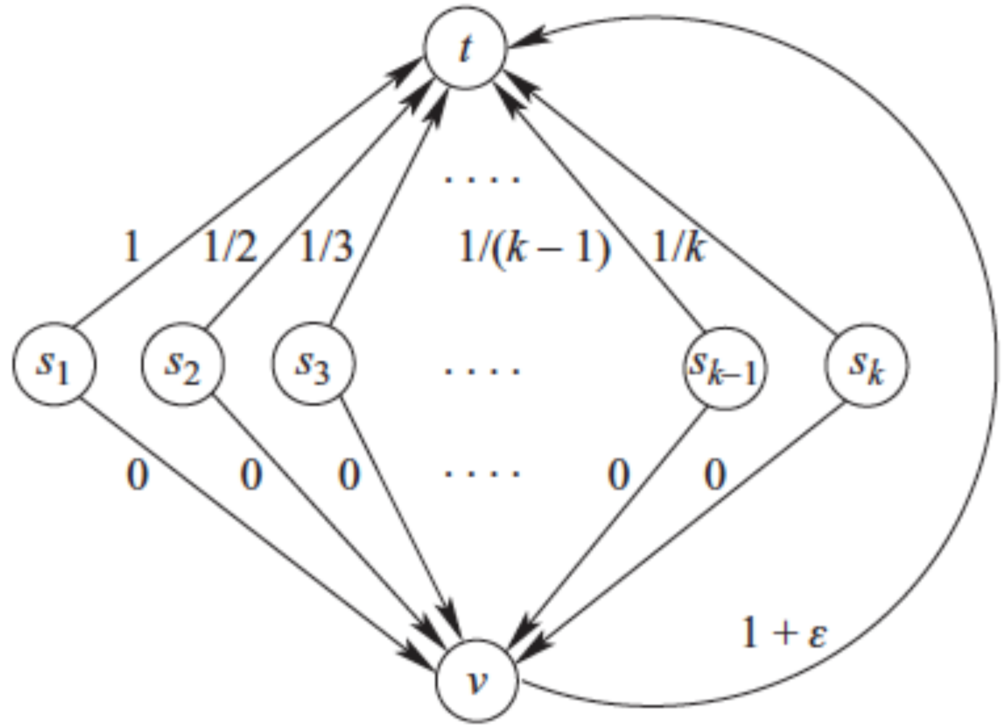
$$\frac{\text{value of worst equilibrium}}{\text{value of optimal solution}}$$



- Theorem: In any global connection game, the price of anarchy is at most k.
  - Proof:
    - Let S be a stable network and S^* be an optimal network,
    - Let w_i be the weight of the shortest path from s_i to t_i, and let $w(P_i^*)$ be the weight of the path the optimal solution selects for i.
- What about the price of stability?

Price of Stability

- For the given example:
  - What is the optimal solution?
  - What are the equilibria?
  - What is the price of stability?



- Theorem:
  The price of stability is always at most H_k.

- Proof: Potential Method for Games

Price of Stability

- Theorem: A pure Nash equilibrium always exists and the price of stability is at most H_k.

- Potential function method:
  - Define a function on edges $\Phi_e = c_e H_{ke}$ and $\Phi = \Sigma_e \Phi_e$.
  - If i unilaterally changes its strategy to S', then can show that

    $$\Phi(S) - \Phi(S') = u_i(S') - u_i(S)$$

    (this is called a *potential function*).
  - Can also show that:

    $$cost(S) \leq \Phi(S) \leq H_k \, cost(S)$$

- Theorem: A strategy that minimizes Φ is stable.

- Theorem: If $A \, cost(S) \leq \Phi(S) \leq B \, cost(S)$, then the price of stability is at most B/A.

Co-Author Game

Coauthor Game

- Coauthor game (buisness/collaboration networks)
  - Agents can partner together, but the more partners an agent has the less resources she has to put into the partnership.
- Each of the n agents is a node.
  - Nodes benefit from partnerships (direct edge connections) to others due to "collaboration".
  - The amount a node benefits is inversely proportional to the amount of partnerships (i.e., degree) one has.

Coauthor Game

- Let n_i be the degree of node i
- Node *i* wishes to maximize:

$$u_i(g) = \sum_{j:\, ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right]$$

  - If $n_i = 0,\ u_i(g) = 0$

- A network is optimal if it maximizes:

$$\sum_{i \in N} u_i(g) = \sum_{i:\, n_i > 0} \sum_{j:\, ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right],$$

Optimal Networks

- A network is optimal if it maximizes:

$$\sum_{i \in N} u_i(g) = \sum_{i:\, n_i > 0} \sum_{j:\, ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right],$$

- Goal: Find an optimal network:

- Approach:
  - First, upper bound $\sum_{i \in N} u_i(g)$

    $$\sum_{i \in N} u_i(g) \leqslant 3\dot{N}$$

  - Show some network attains this bound:
    - An optimal network on 2K agents consists of K pairs of nodes.

- Is this network is an equilibrium?
  - No

Stable Networks

- Node *i* wishes to maximize:

$$u_i(g) = \sum_{j:\, ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right] = 1 + \left(1 + \frac{1}{n_i}\right) \sum_{j:\, ij \in g} \frac{1}{n_j},$$

- Node i would like to link to node j if:

$$\frac{n_i + 2}{n_j + 1} > \frac{1}{n_i} \sum_{k:\, k \neq j,\, ik \in g} \frac{1}{n_k}.$$

- Assume that $n_j \leqslant n_i$,
  - Does i want to link to j?
    - Yes! What does this mean for stable networks?
- The only stable networks are complete networks.

Price of Stability / Price of Anarchy

- Recall: The optimal network has cost

$$\sum_{i \in N} u_i(g) \leqslant 3N$$

- Recall: The only stable networks are complete networks.

- The price of stability is: $\frac{\text{value of best equilibrium}}{\text{value of optimal solution}}$
  - Is > 1/3

- The price of anarchy is: $\frac{\text{value of worst equilibrium}}{\text{value of optimal solution}}$
  - In this case, is the same!

Does this notion of stability make sense?

- We showed conditions under which i wants to connect to j. Does j also want to connect to i?
  - In some cases, the notion of stability may be incomplete.

- Pairwise stability:
  - For all ij in g,we have $u_i(g) > u_i(g\text{-}ij)$ and $u_j(g) > u_j(g\text{-}ij)$.
  - For all ij not in g,we have $u_i(g) < u_i(g+ij)$ and $u_j(g) < u_j(g+ij)$.

- Theorem: The pairwise-stable networks can be decomposed into fully connected components with no cross-component edges such that if the number nodes in each of the t components is $k_1 > k_2 > \ldots > k_t$, then $k_{i-1} > k_i^2$.

- Proof: Similar analysis as before, but show that j in the smaller component only wants to connect to i if the above is satisfied.

Complements vs Substitutes

- In the local connection game, the more connections other agents build, the fewer connections we build.
  - This is a game of *strategic substitutes.*

- In the coauthor game, the more connections other agents build, the more connections we build.
  - This is a game of *strategic complements.*

Price of Anarchy

- The Price of Anarchy is the ratio: $\dfrac{\text{value of worst equilibrium}}{\text{value of optimal solution}}$
- Lemma:
  - If $\alpha \geq 2$, then the optimal network is a star.
  - If $\alpha < 2$, then the optimal network is a complete graph.
- Lemma:
  - If $\alpha \geq 1$, then any star is a Nash equilibrium.
  - If $\alpha \leq 1$, then the complete graph is a Nash equilibrium.
- Theorem: The price of anarchy is at most $O(\sqrt{\alpha})$.
  - Bound the diameter of an equilibrium graph.
  - Use this to bound the price of anarchy.

Price of Anarchy

- An optimal network minimizes:

$$\alpha|E| + \Sigma_{u \neq v} d(u,v)$$

- Theorem: The price of anarchy is at most $O(\sqrt{\alpha})$.
  - Bound the diameter of an equilibrium graph.
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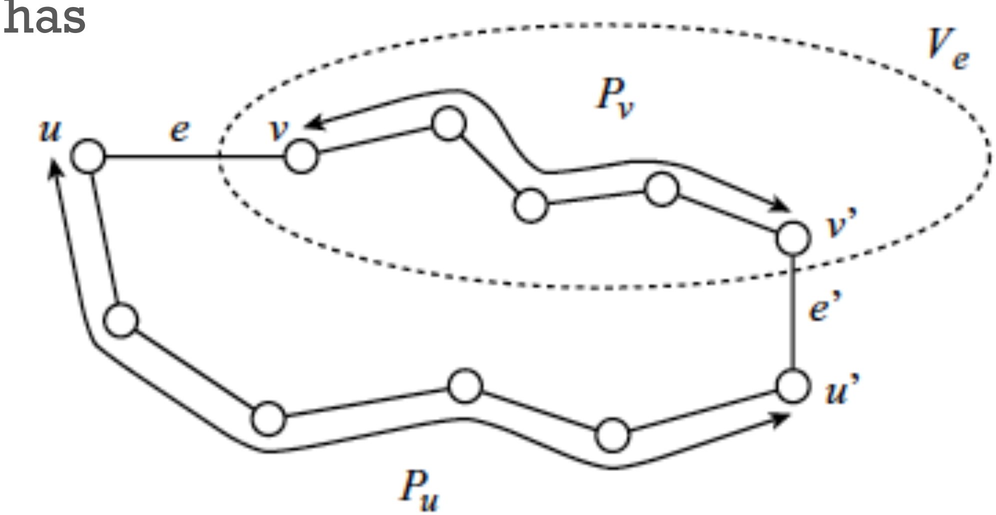
- Lemma: If a Nash equilibrium has diameter d, then its cost is at most O(d) times optimal.



Price of Anarchy

- Theorem: The price of anarchy is at most $O(\sqrt{\alpha})$.
  - Bound the diameter of an equilibrium graph.
  - Use this to bound the price of anarchy.
- Use Previous Lemma: If a Nash equilibrium has diameter d, then its cost is at most O(d) times optimal
- Lemma: The diameter of a Nash equilibrium is at most $2\sqrt{\alpha}$.
- Theorem: Price of anarchy is O(1) when α is $O(\sqrt{n})$.