QUANTUM PHYSICS III

Problem Set 1 21 September 2018

1. Gaussian Integrals

Gaussian integrals are of particular interest in physics. They are intensively used in quantum mechanics and further related topics. Calculate

$$I_1 = \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \ , \quad I_2 = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+bx} \ , \quad I_3 = \int_{-\infty}^{\infty} dx\, x^2\, e^{-\frac{x^2}{2}} \ . \tag{1}$$

2. A Gaussian packet

Consider the following Cauchy problem,

$$-\frac{\hbar}{i}\frac{d\Psi(p,t)}{dt} = H\Psi \ , \quad H = \frac{p^2}{2m} \ , \quad \Psi(p,0) = \frac{A}{(2\pi)^{1/4}} e^{-\frac{\sigma^2}{\hbar^2}(p-p_0)^2} \ , \tag{2}$$

where we adopted the notations

$$\Psi(p,0) = \langle p|\Psi\rangle \ , \quad \Psi(p,t) = \langle p|\Psi(t)\rangle \ , \tag{3}$$

and $|\Psi\rangle$ denotes the state of the quantum mechanical system.

1. Determine the normalization constant A.
2. Find the Fourier image $\Psi(x,0)$ of $\Psi(p,0)$. Compute the dispersions $\Delta x(0)$ and $\Delta p(0)$ of the operators $x(0)$ and $p(0)$ in the state $|\Psi\rangle$.
3. Find $\Psi(p,t)$ and $\Psi(x,t)$. Write the expressions for the dispersions $\Delta x(t)$, $\Delta p(t)$ and describe their behaviour. What can one say about the quantity $\Delta x(t)\Delta p(t)$ at $t = 0$ and $t > 0$?

3. Quantum fluctuations

Consider a hill with a flat top of length $l = 1\ cm$ at the center of which placed an object of mass $m = 1\ g$, whose size is smaller than l.

1. Describing the object by a Gaussian packet, find how long it can stay on the top before quantum fluctuations drive it out. Assume no interaction between the object and the hill.
2. Compute this time in the case when the dispersion of the object is $\sigma = 10^{-9}\ cm$.

4. Harmonic oscillator

Let H be a Hamiltonian of the harmonic oscillator, and a, $a^\dagger$ and $a(t)$, $a^\dagger(t)$ are its annihilation and creation operators in the Schroedinger and Heisenberg pictures correspondingly.

1. Calculate $[a, H]$, $[a^\dagger, H]$, and $[a(t), a^\dagger(t)]$.
2. Find the expression for H in terms of classical variables $\alpha(t)$ and $\alpha^*(t)$ with $\alpha(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x(t) + i\sqrt{\frac{1}{\hbar m\omega}}p(t)\right)$.

5. Gaussian integrals in more dimensions

Consider an $N \times N$ real positive definite symmetric matrix A and two N-dimensional vectors x and B.

1. Show that

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1 ... dx_N \; e^{-\frac{1}{2}x^t A x + B^t x} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \exp\left(\frac{1}{2}B^t A^{-1} B\right) . \tag{4}$$

   Note : The result also holds for the complex symmetric A with the positive definite Re A and arbitrary complex vector B (no conjugation !). The generalization of these integrals to infinite dimensional space is the *path integral*. You will encounter it soon.

2. Consider the “gaussian average” (or *correlator*) defined as

$$\langle x_{i_1} x_{i_2} ... x_{i_p} \rangle = \frac{\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1 ... dx_N \; e^{-\frac{1}{2}x^t A x} x_{i_1} ... x_{i_p}}{\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1 ... dx_N \; e^{-\frac{1}{2}x^t A x}} . \tag{5}$$

   Show that

$$\langle x_{i_1} x_{i_2} \rangle = \left(A^{-1}\right)_{i_1 i_2} ,$$

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle = \langle x_{i_1} x_{i_2} \rangle \langle x_{i_3} x_{i_4} \rangle + \langle x_{i_1} x_{i_3} \rangle \langle x_{i_2} x_{i_4} \rangle + \langle x_{i_1} x_{i_4} \rangle \langle x_{i_2} x_{i_3} \rangle , \tag{6}$$

$$\langle x_{i_1} x_{i_2} ... x_{i_k} \rangle = 0 , \quad \text{if } \; k \; \text{ is odd.}$$

   Note : The formula (6) is an example of the so-called *Wick theorem* : the expectation value is obtained by summing over all possible pairwise contractions. Note also that inserting into the exponent in eq. (5) the higher order terms in x would break this nice relation between the different correlators.