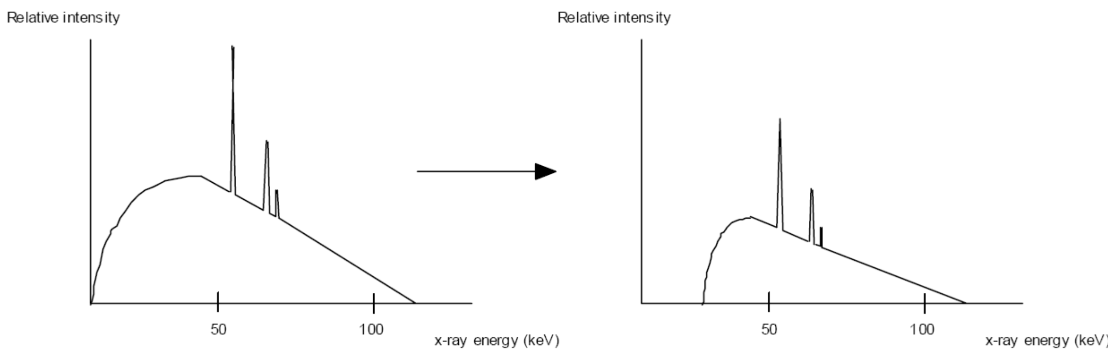
Solution 1 – Beam Hardening

a) Since higher energy X-rays are attenuated less than lower energy X-rays as they pass through tissue, the energy spectrum will shift to higher values. Very low energy X-rays will be almost completely attenuated, as shown below.



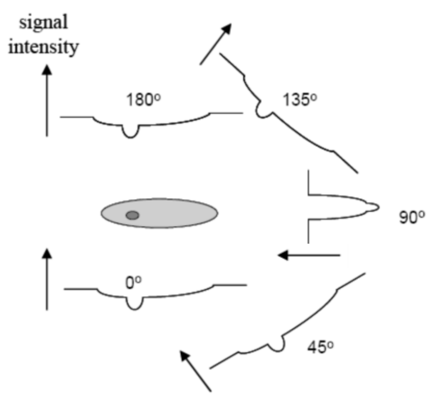
b) Top beam $I = I_0 e^{-(10\times1+1\times3)} = I_0 e^{-13} = 2.26\times10^{-6} I_0$

Middle beam $I = I_0 e^{-(10\times3+1\times1)} = I_0 e^{-31} = 3.44\times10^{-14} I_0$

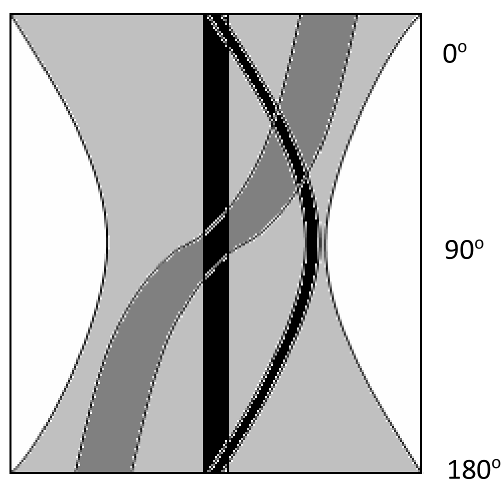
Bottom beam $I = I_0 e^{-(1\times4)} = I_0 e^{-4} = 1.83\times10^{-2} I_0$

Solution 2 – Sinogram

a)



b)



Solution 3 – Reconstruction Methods

a) With $F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $p = \begin{pmatrix} 3 \\ 7 \\ 4 \\ 5 \end{pmatrix}$ we find $a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

b) The back-projection intermediary step b_{int} is given by

$$b_{int} = \begin{pmatrix} 7+6 & 7+4 \\ 6+3 & 4+3 \end{pmatrix} = \begin{pmatrix} 13 & 11 \\ 9 & 7 \end{pmatrix}$$

The normalization is $N = \frac{24+16+22+18}{7+3+6+4} = 4$. Hence we find

$$b_{back-proj} = \begin{pmatrix} 13/4 & 11/4 \\ 9/4 & 7/4 \end{pmatrix}$$

Here $F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $p = \begin{pmatrix} 7 \\ 3 \\ 6 \\ 4 \end{pmatrix}$. We see that $b_{int} = b_2 = F^T p$. This is the very definition of the back-projection. Hence back-projection is equivalent (up to a unimportant scale factor) to approximating the inverse matrix with its transpose (!).

As can be seen on the diagram as well, the measured projections are not independent, for example $P90(1) = P0(0) + P0(1) - P90(0)$. In turns, this means F has zero determinant and hence cannot be inversed. More fundamentally, F has an non-empty kernel spanned by the vector $b_k = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. On the image it means adding any image proportional to $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ won't change the measured projections. Hence we conclude that the problem is ill-posed, and contrast of the reconstructed images is only (maybe) meaning-full on the diagonals, where the impact of any b_k component is the same.

c) Using the minimization method, we find the optimal solution:

$$b_{image} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

Similarly to b): we find $b_{int} = \begin{pmatrix} 18 & 16 \\ 14 & 12 \end{pmatrix}$, $N = \frac{180}{30} = 6$ and

$$b_{back-proj} = \begin{pmatrix} 3 & 8/3 \\ \frac{7}{3} & 2 \end{pmatrix}$$

For exercises b) and c) the results give $r_{b)} = \frac{5}{4}$ and $r_{c)} = \frac{20}{9}$. Hence the solution of exercise b) is closer to the true solution. Adding more projections **is** beneficial when properly taken into account, which the standard back-projection does not do. Using filtered back projection is one way to go. There exist iterative methods (such as conjugate gradient) for finding the solution of the minimization problem in clinically relevant case, where computing the pseudo-inverse matrix is computationally too expensive due to the large number of voxels.

Solution 4 – Spatial Resolution of Back-Projection

1. Thanks to the delta function, this expression amounts to sum all projections passing through the point (x, y), which is by definition the back-projected image.
2. For a given angle φ we have $x = \cos(\varphi)\, x' - \sin(\varphi)\, y'$, $y = \cos(\varphi)\, y' + \sin(\varphi)\, x'$.
   Therefore, we have $\mathrm{g}(x', \varphi) = \int_{-\infty}^{+\infty} \mu(\cos(\varphi)\, x' - \sin(\varphi)\, y', \cos(\varphi)\, y' + \sin(\varphi)\, x')\mathrm{dy}'$.
3. Using a simple change of variable we find $\int f(O\vec{r}')e^{i\vec{k}\cdot\vec{r}'}\mathrm{d}^2r' = \int f(\vec{r})e^{iO\vec{k}\cdot\vec{r}}\mathrm{d}^2r = \tilde{f}(O\vec{k})$.
   This is the central slice theorem. It comes from combining the results of 2. and 3. with $k_y = 0$. This result means that, up to an invertible transformation along x', we are actually measuring the Fourier transform of μ on radial spokes.
4. We have $\mu_B(x, y) = \int_{-\infty}^{+\infty} \int_0^{\pi} \mathrm{g}(x', \varphi) \int_{-\infty}^{+\infty} e^{ia(\mathrm{x}' - (\cos(\varphi)\mathrm{x} + \sin(\varphi)y))} \mathrm{da}\, \mathrm{d}\varphi \mathrm{d}x'$ from 1.
   Integrating over $x'$and using the result of 4 we get
   $$\mu_B(x, y) = \int_{-\infty}^{+\infty} \int_0^{\pi} \tilde{\mu}(\cos(\varphi)\, a, \sin(\varphi)\, a)\, e^{-\mathrm{ia}(\cos(\varphi)\mathrm{x} + \sin(\varphi)y)}\, \mathrm{da}\, \mathrm{d}\varphi$$
   The change of variable $k_x = \cos(\varphi)\mathrm{a}$, $k_y = \sin(\varphi)\mathrm{a}$ gives the desired result.
5. From point 3., we see that the Fourier transform of $\frac{1}{|\vec{k}|}$ has to be invariant under rotation, hence it is only a function of $|\vec{x}|$. Its dimension is $\left[\frac{1}{|\vec{k}|}\right] [\vec{k}]^2 = [\vec{k}] = \left[\frac{1}{|\vec{x}|}\right]$ and hence it is proportional to $\frac{1}{|\vec{x}|}$.
6. The back-projected image is the true image convolved with $\frac{1}{|\vec{x}|}$, which produces blurring.
   Filtered back projection amounts to applying the Fourier transform to the function g (as in exercise 4.4), multiplying by $|\vec{k}|$, and applying the inverse Fourier transform. This defines a filtered version of the function g on which we can apply the usual back-projection procedure.