9: Relaxation of nuclear magnetization

1. How is the MR signal detected ?
2. What is the quantum-mechanical equivalent of the rotating frame ?
3. What is the rotating frame description good for ?
4. How can the return of the magnetization to thermodynamic equilibrium described ?
5. How is the time-dependent change of magnetization described mathematically ?

After this course you
1. Can describe the principle of MR detection and excitation
2. Can explain how MR excitation is frequency selective (resonance)
3. Understand the principle of relaxation to the equilibrium magnetization
4. Know what are the major relaxation times and how they phenomenologically affect magnetization in biological tissue, in particular that of water.
5. Can explain the elements of the Bloch equations and FID
6. Understand the MR contrast strongly depends on experimental parameters

What do we know about MR so far ?

Need:
Nucleus with non-zero spin
Magnetic field B_0

Get:
Nuclear (equilibrium) magnetization M_0
(Magnitude dictated by Boltzmann distribution)
M_0 increases with
1. Number of spins in voxel
2. Magnetic field B_0
3. Gyromagnetic ratio γ

Imaging ^1H in H_2O is most sensitive

Thermodynamic equilibrium magnetization M_0 is || B_0

$$\frac{d\vec{M}_0}{dt} = \vec{M}_0 \times \gamma \vec{B}_0 = 0$$

M_0 does not precess

⇨ All this does not generate a measurable signal

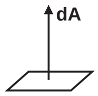
9-1. How is the MR signal detected ?

Faraday's Law of Induction

$$\xi = -\frac{d\Phi_B}{dt}$$

Magnetic flux Φ_B

$$\Phi_B = \iint_{surface} \vec{B}(\vec{r},t)\cdot d\vec{A}$$



MR: dA=constant

$\rightarrow \xi \propto dB/dt$

Lenz's Law
induced voltage $\xi \Rightarrow$ current $\rightarrow$ magnetic field opposes the change in the magnetic flux that produces the current (Completely analogous to power generation!)

Biot-Savart Law
magnetic field falls off with r^{-2} $\vec{B}(\vec{r}) \propto \int \frac{\mu_0}{4\pi} I \frac{d\vec{l}\times\vec{r}}{r^3}$

Magnetic field of dipole decreases with distance :
ξ **decreases with distance from magnetization**

9-2. Rotating frame revisited

Equation of motion for M (always valid in any reference frame) in presence of B_0

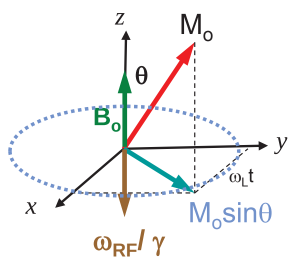
$$\frac{d\vec{M}}{dt} = -\gamma\Delta\vec{B}^{eff}\times\vec{M}$$

magnetization **precesses** in xy plane with frequency $\gamma\Delta B^{eff}/2\pi$

Rotating frame: reference frame rotating about z at frequency ω_{RF}

Case I: non-rotating reference frame (ω_{RF}=0)

$\Rightarrow$ magnetization **precesses** in xy plane with frequency $\gamma B_0/2\pi$



Case II: rotating frame with $\omega_{RF} = \omega_L$

$\Rightarrow$magnetization is **stationary** ("precesses" in xy with **zero** frequency)

Equation of motion is still valid, i.e. precession frequency $\gamma\Delta B^{eff}/2\pi$

$\Rightarrow \Delta B^{eff} = 0$

Larmor frequency Ω in the rotating frame:

$\Omega = \gamma\Delta B^{eff}$

$\Delta B^{eff} = B_o - \omega_{RF}/\gamma$

Supplement: Rotating frame

What are the quantum-mechanical equivalencies ?

Schrödinger representation:

$$i\hbar \frac{d}{dt}\left|\psi_S(t)\right\rangle = H_S\left|\psi_S(t)\right\rangle$$

If H_S=const in t: $\left|\psi_S(t)\right\rangle = e^{-iH_S t/\hbar}$

NB. $\left\langle I_z \right\rangle \equiv \left\langle \psi_S(t)\right| I_z \left|\psi_S(t)\right\rangle$

Quantum mechanical equivalencies:
$M_z \propto \langle I_z\rangle$, $M_x \propto \langle I_x\rangle$, $M_y \propto \langle I_y\rangle$

For one spin-1/2 (^{1}H), i.e. two energy levels

$$I_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad I_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

How to determine $\langle I_x(t)\rangle$ etc ?
⇒ Split H_S into time-invariant and -dependent terms:

$$i\hbar \frac{d}{dt}\left|\psi_S(t)\right\rangle = \left[H_S^{\,0} + V(t)\right]\left|\psi_S(t)\right\rangle$$

Interaction representation
(Higher order perturbation theory)

$$\left|\psi_I(t)\right\rangle \equiv e^{iH_S^{\,0} t/\hbar}\left|\psi_S(t)\right\rangle$$

$$i\hbar \frac{d}{dt}\left|\psi_I(t)\right\rangle = V_I(t)\left|\psi_I(t)\right\rangle$$

$$V_I(t) = e^{iH_S^{\,0}/\hbar} V_S(t) e^{-iH_S^{\,0}/\hbar}$$

For spin: $H_S^{\,0} = \hbar\gamma B_0 I_z$

$$V(t) = \hbar\gamma B_1\left(\cos(\omega_{RF} t) I_x + \sin(\omega_{RF} t) I_y\right)$$

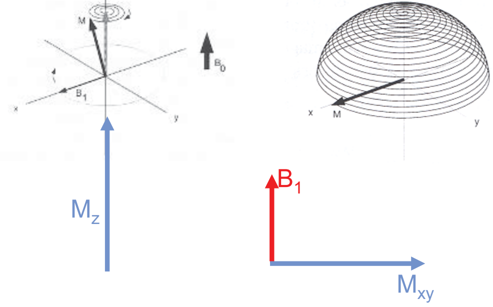
What is $V_I(t)$ [$\omega_{RF}=\gamma B_0$]? $V_I(t) = \hbar\gamma B_1 I_x$

Quantum mechanical equivalencies:
$B_0 \propto I_z$, $B_{1x,y} \propto I_{x,y}$

9-3. What is the motion of magnetization when an RF field induces a flip angle ?

Laboratory frame of reference

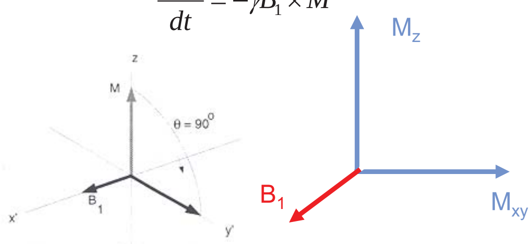
$$\frac{d\vec{M}}{dt} = -\gamma\left(\vec{B}_1(t) + \vec{B}_0\right)\times\vec{M}$$



B_1 radiofrequency field at Larmor frequency ω_L applied in transverse (*xy*) plane for duration τ
⇒ **nutation** (at ω_L) of M as it tips away from the *z*-axis.

Rotating frame of reference

$$\frac{d\vec{M}}{dt} = -\gamma\vec{B}_1 \times \vec{M}$$



RF field rotates M towards xy plane

Amplitude B_1 determines how quickly the magnetization is rotated.

flip angle $\alpha = \gamma B_1 \tau$ [rad]

$$M_z = M_0 \cos\alpha$$
$$M_{xy} = M_0 \sin\alpha$$

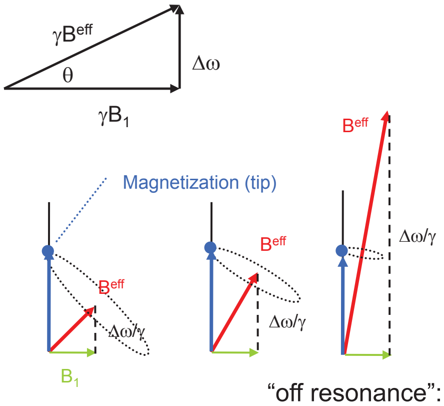
In MRI typically $\gamma B_1/2\pi$ ~ 0.1-1kHz (τ~1ms)

What is “resonance” ?

What range of frequencies can be excited with a given RF pulse?

At $\Delta\omega=\omega_L-\omega_{RF}$ (from ω_L) magnetization experiences effective field strength B^{eff}

$$\gamma B^{eff} = \sqrt{(\gamma B_1)^2 + (\Delta\omega)^2}$$



Rotation axis : tilted by θ.

“on resonance”:

$\gamma B_1 >> \Delta\omega \rightarrow$ effective field || B_1

$\Rightarrow$ short RF pulses (τ<1ms)

“off resonance”:

$B_1 << \Delta B \rightarrow B^{eff}$ || z

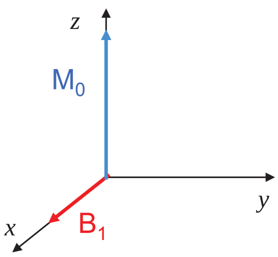
RF field with amplitude B_1 can excite a range of frequencies on the order of $\pm\gamma B_1$

Quantum mechanical “resonance”
Transition probability highest : $h\nu = h\gamma B_0/2\pi$

9-4. How is the return to equilibrium M_0 governed ?

Relaxation

Thermodynamic equilibrium

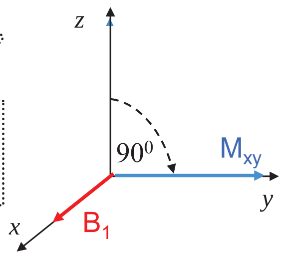


RF pulse(s) B_1

Relaxation

T_1, T_2

After excitation



Transverse magnetization:

(along x and y-axis, on resonance)

$$\frac{dM_x(t)}{dt} = -\frac{M_x(t)}{T_2} \qquad \frac{dM_y(t)}{dt} = -\frac{M_y(t)}{T_2}$$

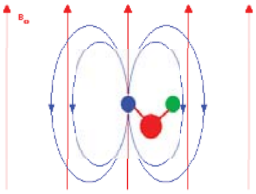
Equations formally equivalent to linear attenuation coefficient (x-ray) (same solution)

Exponential decay of M_{xy}

$$M_{xy}(t) = M_{xy}(0)e^{-\frac{t}{T_2}}$$

What are the mechanisms of relaxation ?

Tumbling of Molecule
(Brownian motion)
Creates local oscillating/fluctuating
magnetic field

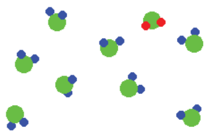


Fluctuating magnetic field

depends on orientation of the whole molecule & correlation time τ_c (=time for molecule to rotate 1 rad)

Sources of fluctuating magnetic field:

- Dipolar coupling between nuclei and solvent
- interaction between nuclear magnetic dipoles



Correlation function G $$G(t) \propto e^{-t/\tau_c}$$

Describes degree of correlation of motion t sec apart

Correlation time τ_c: $$\tau_c = \frac{4\pi\eta r^3}{3kT}$$

η : viscosity
k: Boltzmann constant
r: size of molecule

What is the cause of loss of transverse Magnetization ?

fluctuating *microscopic* magnetic fields δB

T_2: phenomenological time constant

Range 10µs (bone)… several s (water)

“transverse relaxation”, “T2 relaxation”

Cause:

Molecular dynamics and spin-spin interactions

Historically : “spin-spin” relaxation

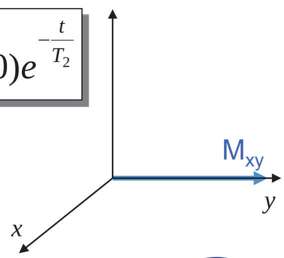
→ loss of signal in xy plane

“Memory” relaxation

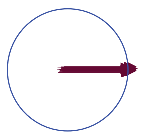
Rule of thumb for tissue water:
The less “tissue” (bone, solutes, proteins, membranes) is in contact with bulk water, the longer bulk water T_2

$$M_{xy}(t) = M_{xy}(0)e^{-\frac{t}{T_2}}$$

After excitation



random precession of nuclei
→ dephasing of spins with time constant ‘T_2’



Phase ϕ accrued over τ_c:
$\phi=\delta B \tau_c$

$$\iiint_{voxel} \rho(\vec{r}) e^{i\delta B(\vec{r},t)\tau_c} dV \to 0$$

τ_c large (immobile spins):

large phase differences ⇒ short T_2

bone, membranes, proteins are MR-”invisible”

How does M_z return to equilibrium ?

Longitudinal relaxation T_1

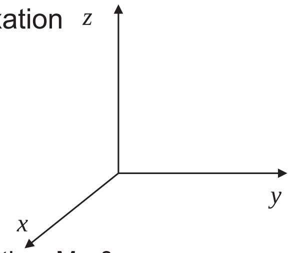
After decay of M_{xy} by T_2: $M_z \rightarrow M_0$

Longitudinal Relaxation

(along z-axis) $\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_0}{T_1}$

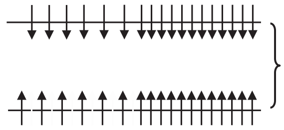
$$M_z(t) = M_0(1 - e^{-t/T_1}) + M_z(0)e^{-t/T_1}$$

After T_2 relaxation



After 90^0 excitation: M_z=0

⇒ population distribution corresponds to T=∞:



m=-1/2

$\Delta E = \gamma \cdot \frac{h}{2\pi} \cdot B_0$

m=1/2

$\frac{N_1}{N_2} = e^{-\frac{\Delta E}{kT}}$

Mechanisms: Incoherent molecular fluctuations on the order of the Larmor frequency ω_L
possibility of energy transfer → matching frequency

Historically : spin-lattice relaxation
(heat lost to the surroundings)
T_1 ~ 0.5-5s (water)

> Rule of thumb for water:
> The less “tissue” is in contact with bulk water (bone, solutes, proteins, membranes), the longer bulk water T_1

Boltzmann distribution re-established by energy (photon) transfer from spins to system (lattice).

Most efficient when energy levels of system and nuclear spins match, i.e.

$\omega\tau_c$~1 ⇒ T_1 minimal

(bone: $\omega\tau_c$>>1 ⇒ T_1~ s to min)

9-5. What equations describe the change in magnetization ?

Bloch Equations

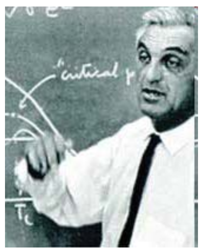
add relaxation terms (T_1, T_2) to the fundamental Eq of motion of magnetization:

$$\frac{dM_z(t)}{dt} = \gamma[M_x(t)B_y(t) - M_y(t)B_x(t)] - \frac{M_z(t) - M_0}{T_1} \quad \text{along z}$$

$$\frac{dM_x(t)}{dt} = \gamma[M_y(t)B_z(t) - M_z(t)B_y(t)] - \frac{M_x(t)}{T_2} \quad \text{along x}$$

$$\frac{dM_y(t)}{dt} = \gamma[M_z(t)B_x(t) - M_x(t)B_z(t)] - \frac{M_y(t)}{T_2} \quad \text{along y}$$

$$-\gamma\vec{B} \times \vec{M}$$



Felix Bloch
Physics
1952

Substituting Ω=-γB_0+ω_{RF} (B_0=B_z is not time-dependent) yields:

Rotating reference frame

$$\gamma[M_x(t)B_1^y(t) - M_yB_1^x(t)]$$

$$-\Omega M_y(t) - \gamma M_z B_1^y(t)$$

$$\gamma M_z(t)B_1^x(t) + \Omega M_x$$

$$(\gamma\vec{B}_1 + \vec{\Omega}) \times \vec{M}$$

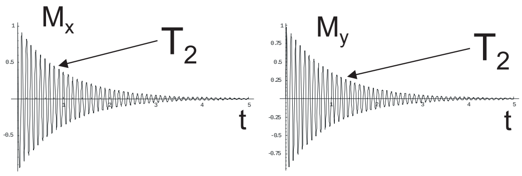
B_1 : RF field in rotating frame

What characterizes the basic MR signal ?

Free induction decay: Precession and relaxation (after RF pulse)

Transverse magnetization

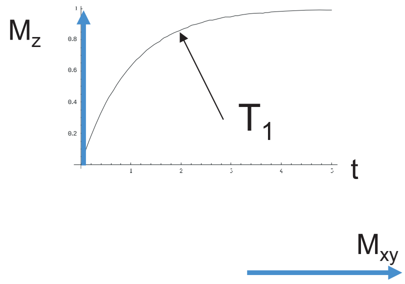
$$M_{xy}(t) = M_{xy}(0)e^{-i\omega t}e^{-t/T_2}$$

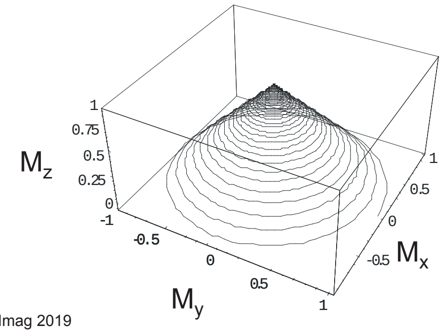


Longitudinal magnetization

(after 90^0 RF excitation)

$$M_z(t) = M_0(1 - e^{-t/T_1})$$



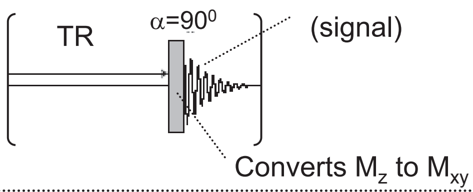


NB. M can never exceed $M_0 \Rightarrow T_2 \leq T_1$

How can T_1 changes be measured ?

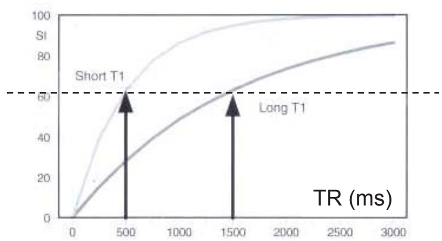
repetitive pulsing

Multipulse experiment with RF pulses applied every TR seconds



$$M_z(TR) = M_0(1 - e^{-TR/T_1})$$

The effect of T_1 (and T_2) on the signal depends on how it is measured



$$M_z(t) = M_0(1 - e^{-t/T_1}) + M(0)e^{-t/T_1}$$

$$M(0) = M_0 \cos\alpha$$

$$\rightarrow M_z(t) = M_0(1 - e^{-t/T_1})$$

Optimal TR to detect changes in T_1 ?

Use noise error propagation calculation (Lesson 1) F=max:

$$\frac{\partial M_z(t)}{\partial T_1} = \frac{t}{T_1^2}e^{-t/T_1} \equiv F \qquad \frac{dF}{dt} = 0 = \frac{1}{T_1^2}e^{-t/T_1} - \frac{t}{T_1^3}e^{-t/T_1}$$

$$0 = \frac{1}{T_1^2}e^{-t/T_1}\left(1 - \frac{t}{T_1}\right)$$ t=TR_{opt}: $TR_{opt} = T_1$

Summary
Magnetic resonance so far

Magnetic field B_0 → Equilibrium magnetization $M_0 || z$ Increases with
1. number of spins in voxel
2. Static magnetic field B_0
3. gyromagnetic ratio γ

RF field B_1 (applied on-resonance i.e. ω_L) → tilts magnetization M into transverse plane xy → Precession of M_{xy} is detected

T_2 and T_1 relaxation → exponential decay of M_{xy}, exp. return of M_z to M_0 → reflect molecular environment; source of contrast

1. Only mobile spins (e.g. water) are detected
2. M_0 reflects amount of nuclei and thus water content
   [Water content varies 70-100ml/100g in body (poor contrast)]
3. Effect of T_1 and T_2 changes on image contrast depend strongly on experimental parameters (RF pulse timing and flip angle)