Final exam—Financial Econometrics

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June 2014

LAST, first name: ____

Hereby I confirm that during the exam I complied with the EPFL honor code

Signature:______________________________

- Write as clearly as possible. Short and precise answers are preferred.
- The percentage for each exercise (grading) is just an indication.
- Hand in the exam in 3 hours.
- The exam is closed book, closed notes.
- Scientific calculators are not allowed.
- You must turn in all sheets at the end of the exam, including any scratch paper. All exam questions must be returned.
- Do not hesitate to use intermediate results!

Exercise 1: Multiple questions ($\simeq$ 25%-30%)

Answer the following questions. **All questions must be justified**.

1. A strongly stationary stochastic process is always a weakly stationary stochastic process:
   <u>Answer</u> :

   ☐ TRUE ☐ FALSE

2. The first partial autocorrelation coefficient of a fundamental AR(p) stochastic process equals the first autocorrelation.
   <u>Answer</u> :

   ☐ TRUE ☐ FALSE

3. Let $(X_t)_{t\geq 0}$ denote the stochastic process defined by:

$$X_t = X_{t-1} + \epsilon_t$$

   where ϵ_t is a weak white noise $(0, \sigma_\epsilon^2)$, and X_0 is a (real-valued) random variable that is uncorrelated with $\epsilon_1, \cdots, \epsilon_t$ (for $t > 0$). Determine $\mathbb{C}ov(X_t, X_s)$.
   <u>Answer</u> :
   ☐ $\mathbb{C}ov(X_t, X_s) = \mathbb{V}(X_0) + t\sigma_\epsilon^2$ for all (t, s)
   ☐ $\mathbb{C}ov(X_t, X_s) = \mathbb{V}(X_0) + s\sigma_\epsilon^2$ for all (t, s)
   ☐ $\mathbb{C}ov(X_t, X_s) = \mathbb{V}(X_0) + (t-s)\sigma_\epsilon^2$ for all (t, s)
   ☐ $\mathbb{C}ov(X_t, X_s) = \mathbb{V}(X_0) + (s-t)\sigma_\epsilon^2$ for all (t, s)
   ☐ None: What is the solution?

4. Suppose that one observes the daily returns of some financial assets. What are the main stylized facts on financial markets for daily returns? Explain carefully.

5. A weak white noise process (ϵ_t) is defined by

   (i) $\mathbb{E}[\epsilon_t] = 0$ for all t;

   (ii) $\mathbb{V}[\epsilon_t] = \sigma_\epsilon^2 < +\infty$ for all t;

   (iii) $\mathbb{E}[\epsilon_t \epsilon_{t+k}] = 0$ for $k > 0$ and t is fixed.

   Answer :

   □ TRUE □ FALSE

6. How can one define historical conditional volatility? Provide definition(s) and comment.

7. Consider the following ARCH(2) specification

$$
\begin{aligned}
r_t &= \mu + \epsilon_t \\
\epsilon_t &= \sigma_t Z_t \\
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2
\end{aligned}
$$

   where r_t is the daily return of a given asset and $Z_t \sim$ i.i.d.$\mathcal{N}(0,1)$.

   (i) Comment the different equations.

   (ii) Determine the equivalent (weak) ARMA(p,q) representation for (ϵ_t^2).

   (iii) Find the conditional variance and unconditional variance of r_t.

8. The daily volatility can be characterized by r_t^2 and $|r_t|$. The two series have the same statistical properties:

   □ TRUE □ FALSE

9. One considers a portfolio with n assets with expected returns $(\mathbb{E}(R_1), \cdots, \mathbb{E}(R_n))^\top$ and variance-covariance matrix Σ. The vector of weights is given by $w = (w_1, \cdots, w_n)^\top$.

   – Write the standard mean-variance portfolio problem.

   – How many inputs does one need to solve the problem? Explain carefully.

   – How can one reduce the number of inputs? Provide a detailed example.

10. Consider three returns (SP&500, DAX and FT) at different frequencies (daily, weekly and monthly). For each return, a GARCH(1,1) model is estimated and Table 1 reports the results. Comment Table 1 (significance of coefficients, adequation with frequency, etc).

Table 1 : Estimation of GARCH models

	S&P500		DAX		FT All Shares	
	statistic	std error	statistic	std error	statistic	std error
Daily returns						
ω	0.0074	0.0012	0.0248	0.0031	0.0166	0.0024
α	0.0513	0.0039	0.0910	0.0065	0.0860	0.0063
β	0.9422	0.0042	0.8954	0.0069	0.8996	0.0069
weekly returns						
ω	0.0829	0.0292	0.2369	0.0634	0.1955	0.0557
α	0.1015	0.0165	0.1091	0.0165	0.1063	0.0177
β	0.8872	0.0174	0.8642	0.0195	0.8570	0.0229
Monthly returns						
ω	0.6531	0.4497	3.4344	1.8789	1.5123	1.0000
α	0.1297	0.0419	0.1276	0.0487	0.1217	0.0466
β	0.8444	0.0505	0.7837	0.0817	0.8101	0.0765

Exercise 2: Approximations ($\simeq$ 15%-20%)

Consider the geometric Brownian motion:

$$\begin{aligned} dX(t) &= \theta_1 X(t)dt + \theta_2 X(t)dW(t) \\ X(0) &= x_0 \end{aligned}$$

where $(W(t))$ is a standard Wiener process. Consider n observations in the interval $[0, T]$ equally spaced at intervals $h \equiv T/n$.

1. Determine the Euler discretization for this stochastic process. Explain carefully.

2. Determine the Milstein discretization for this stochastic process. Explain carefully.

3. Write a Matlab code for the Euler approximation. Explain.

Exercise 3: Modeling a spread ($\simeq$ 15%-20%)

Let (spread_t) denote the (monthly) spread between BAA and AAA corporate bonds over the period 1986m4-2007m4 (Figure 1). Let (dspread_t) denote the first difference of the "spread" variable (Figure 2).

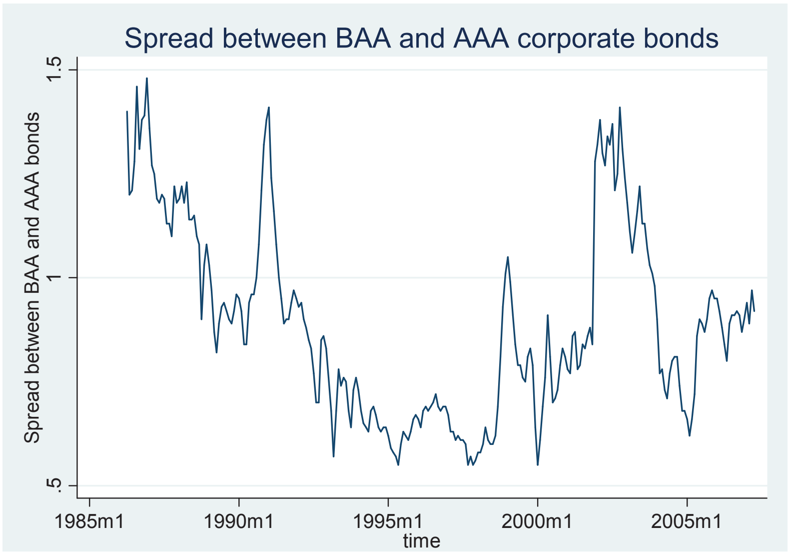

Figure 1:

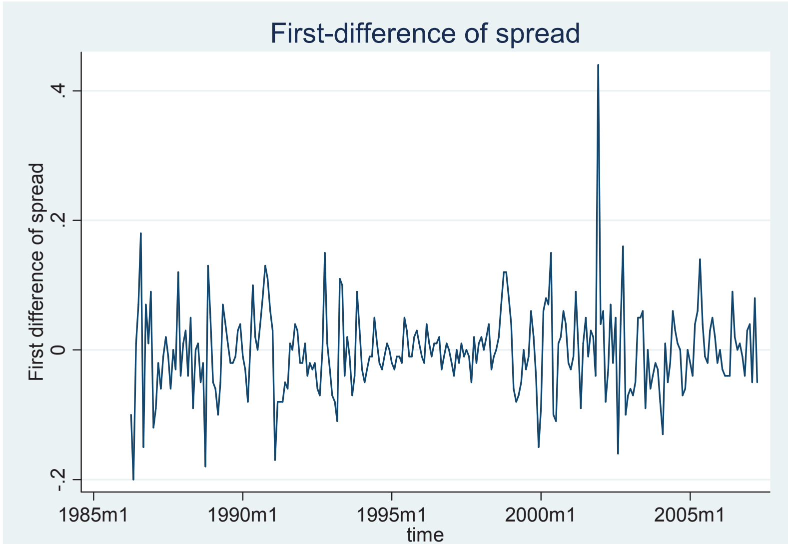

Figure 2:

The autocorrelation function (respectively, partial autocorrelation function) of the variable "spread" is displayed in Figure 3 (respectively, Figure 4) whereas the autocorrelation function (respectively, partial autocorrelation function) of the variable "dspread" is displayed in Figure 5 (respectively, Figure 6). The grey area represents the 95% confidence bands (under the null hypothesis that the (partial) autocorrelation is zero).

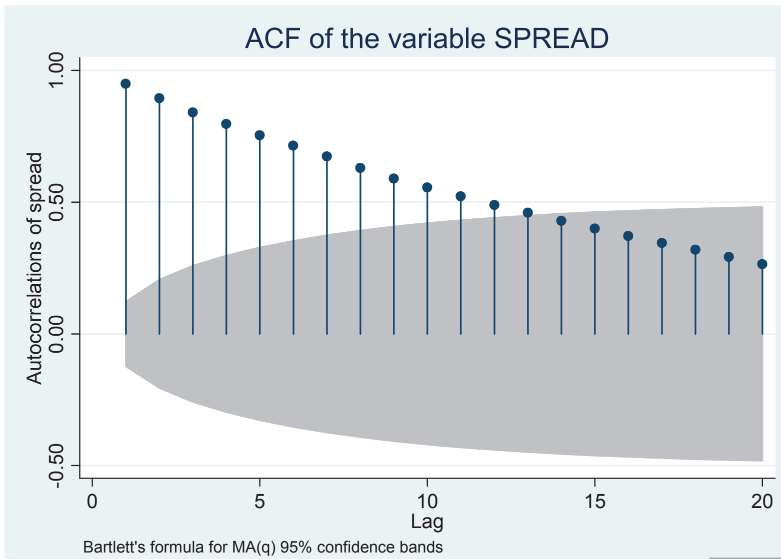

Figure 3:

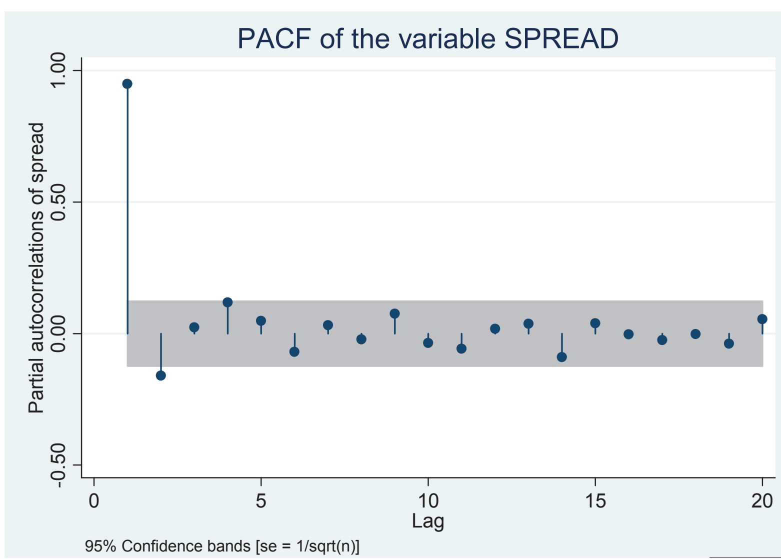

Figure 4:

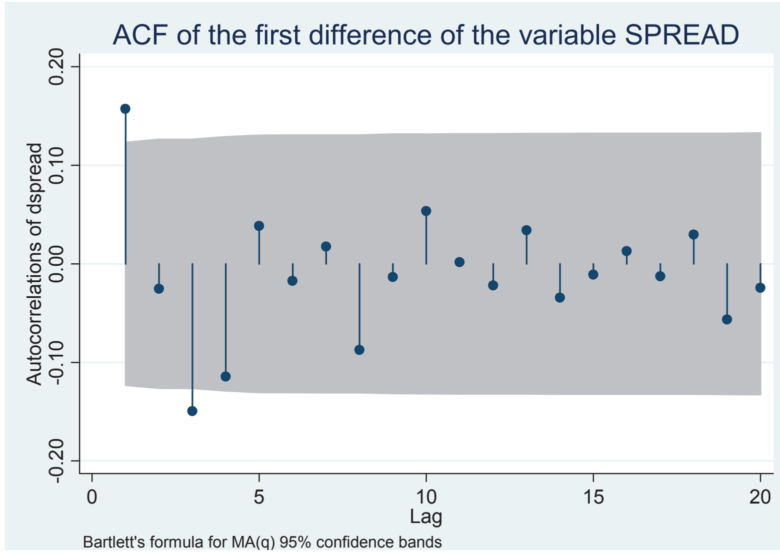

Figure 5:

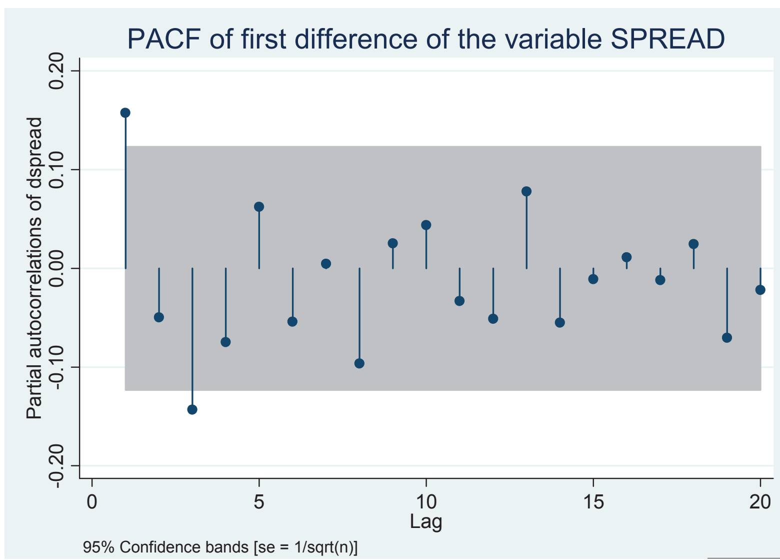

Figure 6:

Table 2: ACF and PACF of the variable "spread"

LAGS	1	2	3	4	5	6	7	8	9	10
ACF	0.949	0.893	0.841	0.796	0.753	0.714	0.673	0.630	0.590	0.556
PACF	0.949	-0.160	0.024	0.119	0.048	-0.069	0.032	-0.022	0.075	-0.036
LAGS	11	12	13	14	15	16	17	18	19	20
ACF	0.522	0.489	0.460	0.429	0.399	0.371	0.345	0.319	0.292	0.265
PACF	-0.05	0.018	0.037	-0.090	0.039	-0.003	-0.025	-0.002	-0.039	0.054

1. Explain the different steps of the Box-Jenkins methodology.

2. Using Figures 1-6 and Table 2, one can conclude that the variable "spread" is nonstationary.

   Answer:

   ☐ TRUE ☐ FALSE

3. Table 3 (respectively, Table 4) provides the information criteria (`AIC` and `SBIC`) in the case of the variable "spread" (respectively, "dspread").

Table 3: Information criteria for the variable "spread"

p \ q	0	1	2	3	4	5
	Panel A. `AIC`					
0	-21.18	-290.11	-432.83	-518.92	-559.83	-584.07
1	-651.32	-656.05	-654.58	-654.88	-655.09	-653.31
2	-656.18	-654.23	-653.02	-653.89	-655.27	-651.45
3	-654.32	-653.82	-655.83	-653.86	-652.24	-656.22
4	-655.96	-654.23	-653.85	-654.45	-652.76	-650.78
5	-654.53	-655.78	-652.33	-652.75	-650.79	-648.95
	Panel B. `SBIC`					
0	-14.11	-279.51	-418.70	-501.25	-538.63	-559.34
1	-640.72	-641.91	-636.91	-633.68	-630.36	-625.04
2	-642.04	-636.57	-631.82	-629.15	-627.00	-619.65
3	-636.65	-632.62	-631.10	-625.59	-620.44	-620.88
4	-634.76	-629.49	-625.59	-622.65	-617.43	-611.91
5	-629.80	-627.51	-620.53	-617.42	-611.93	-606.55

Table 4: Information criteria for the variable "dspread"

p \ q	0	1	2	3	4	5
	Panel A. AIC					
0	-650.72	-655.30	-653.39	-655.32	-657.58	-655.63
1	-655.09	-653.32	-651.75	-655.86	-656.90	-656.15
2	-653.73	-655.79	-656.54	-654.97	-654.08	-654.62
3	-657.06	-655.90	-654.85	-655.25	-654.45	-654.20
4	-656.49	-657.36	-655.85	-654.28	-654.56	-652.62
5	-655.50	-655.93	-655.59	-652.49	-652.62	-650.70
	Panel B. SBIC					
0	-643.65	-644.70	-639.25	-637.66	-636.38	-630.90
1	-644.49	-639.19	-634.08	-634.66	-632.17	-627.88
2	-639.60	-638.12	-635.34	-630.23	-625.81	-622.82
3	-639.39	-634.70	-630.12	-626.98	-622.65	-622.40
4	-635.29	-632.63	-627.58	-622.48	-619.23	-613.75
5	-630.77	-627.66	-623.79	-617.16	-613.75	-608.30

(i) What ie the intuition behind information criteria? What are the main properties of these two information criteria?

(ii) The information criteria of Table 3 and Table 4 can be compared to each other. Justify your answer.

Answer:

☐ TRUE ☐ FALSE

(iii) Which model(s) can be selected? (your answer must be consistent with previous questions).

4. Propose a model for the estimation. How can it be improved using the information provided in the exercise? Justify all of your answers.

Exercise 4: Estimation of a continuous stochastic process ($\simeq$ 30%-35%)

Consider the following trending Ornstein-Uhlenbeck process:

$$dP(t) = (-\theta(P(t) - \mu t) + \mu)dt + \sigma dW(t)$$

where $\theta \geq 0$, $P(0) = p_0$, $t \in [0, +\infty)$, and $(W(t))$ is a Wiener process. This stochastic differential equation can be rewritten:

$$d\left(P(t) - \mu t\right) = -\theta(P(t) - \mu t)dt + \sigma dW(t).$$

Its explicit solution is defined to be:

$$P(t) = \mu t + \exp(-\theta t)p_0 + \sigma \int_0^t \exp(-\theta(t-s))dW(s). \tag{1}$$

Part I

1. Consider n observations in the interval $[0, T]$ equally spaced at intervals $h \equiv T/n$ and let $p_0 = 0$. Let P_k denote $P(kh)$ for $k = 0, \cdots, n$. Let ρ denote $\rho = exp(-\theta h)$. Show that Eq. 1 or 1 can be written as:

$$P_k - \rho P_{k-1} - \mu h\left(k - \rho(k-1)\right) = \sigma_\epsilon \varepsilon_k \tag{2}$$

   where $\varepsilon_k \sim$ i.i.d.$\mathcal{N}(0,1)$ and

$$\sigma_\varepsilon^2 \equiv \mathbb{V}\left[\sigma \int_{(k-1)h}^{kh} \exp(-\theta(kh - s))dW(s)\right] = \frac{\sigma^2}{2\theta}\left(1 - \exp(-2\theta h)\right).$$

2. Taking the maximum likelihood estimator of μ, ρ and σ_ε^2 (say, $\hat{\mu}$, $\hat{\rho}$ and $\widehat{\sigma}_\varepsilon^2$), How can we obtain the maximum likelihood estimate of θ and σ^2? Explain carefully.

   Hint: Remember the definition of ρ.

3. We are now interested in the maximum likelihood estimator of $(\mu, \rho, \sigma_\varepsilon^2)^\top$. Using Eq. 2, write the likelihood function and then the log-likelihood function.

4. Write the necessary first-order conditions.

5. Provide a solution for $(\hat{\mu}, \hat{\rho}, \widehat{\sigma}_\varepsilon^2)^\top$.

   Remark: The estimate of one parameter might depend on the other estimates. For instance, $\hat{\mu}$ might depend on $\hat{\rho}$.

Part II Suppose now that the trend μ is known exactly.

6. Propose an estimate of $(\hat{\rho}, \widehat{\sigma}_\varepsilon^2)^\top$.

7. Determine the second-order conditions.

8. Determine the asymptotic variance-covariance matrix. Notably show that

$$\lim_{n \to \infty} - \mathbb{E}\left[\frac{1}{n} \frac{\partial^2 \ell}{\partial \rho \partial \sigma_\varepsilon^2}(.)\Big|_{(\hat{\rho}, \widehat{\sigma}_\varepsilon^2)} \right] = 0$$

   where ℓ is the log-likelihood function.

9. Using the previous question, show that

$$\begin{aligned} \sqrt{n}\,(\hat{\rho} - \rho) &\xrightarrow{d} \mathcal{N}(0, 1 - \exp(-2\theta h)) \\ \sqrt{n}\left(\widehat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2\right) &\xrightarrow{d} \mathcal{N}(0, 2\sigma_\varepsilon^4). \end{aligned}$$

10. Conclude that (using Questions 2 and 9):

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{h^2}\left(\exp(2\theta h) - 1\right)\right).$$

Part III Suppose now that $\mu = 0$.

11. Show that the estimator of ρ and thus θ can be found by minimizing a quadratic function.

12. Show that the corresponding estimator of θ is defined to be:

$$\hat{\theta} = -\frac{1}{h} \ln\left(\frac{\sum_k P_{k-1} P_k}{\sum_k P_{k-1}^2} \right).$$