Final exam — Financial Econometrics

LAST, first name: ____

17 June 2015

Hereby I confirm that during the exam I complied with the EPFL honor code

Signature:______________________________

- Write as clearly as possible. When required, answer in one sentence.
- The exam is closed book, closed notes.
- Hand in the exam in 3 hours.
- The final exam is organized as follows:
  - Exercise 1 (Theory): Autoregressive models (p. 2);
  - Exercise 2: Questions related to the course (pp. 3-5);
  - Exercise 3: Interpretation (pp. 6-13).
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Exercise 1: Autoregressive models

1. State the conditions for a time series to be covariance stationary.

2. Let $(X_t, t \in \mathbb{Z})$ be an AR(1) process given by

$$X_t = \mu + \phi_1 X_{t-1} + \epsilon_t$$

where $|\phi_1| < 1$ and $\epsilon_t \sim i.i.d.(0, \sigma_\epsilon^2)$. Show that $(X_t, t \in \mathbb{Z})$ is covariance stationary by verifying the conditions that you stated in Question 1.

3. Consider and ARMA(p,q) process with lag polynomials $\Phi_p(L)$ and $\Theta_q(L)$. Characterize the conditions for covariance stationarity in terms of the roots of the characteristic equation associated with $\Phi_p(L)$.

4. Now, let $(X_t, t \in \mathbb{Z})$ be an AR(2) process given by

$$X_t = \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

where $\epsilon_t \sim i.i.d.(0, \sigma_\epsilon^2)$. Assume that $\phi_1^2 + 4\phi_2 \geq 0$ and find the roots of the characteristic equation associated with the autoregressive lag polynomial. In the sequel, these roots are denoted z_+ and z_-, with $z_+ \geq z_-$.

5. Still assuming $\phi_1^2 + 4\phi_2 \geq 0$, show that $z_+ < 1$ implies $\phi_1 + \phi_2 < 1$ and $-1 < z_-$ implies $\phi_2 - \phi_1 < 1$. Conclude that if the AR(2) process in Question 4 is covariance stationary then $\phi_1 + \phi_2 < 1$ and $\phi_2 - \phi_1 < 1$ and $-1 < \phi_2 < 1$.

6. In fact, one can show that an AR(2) process is covariance stationary if and only if $\phi_1 + \phi_2 < 1$ and $\phi_2 - \phi_1 < 1$ and $-1 < \phi_2 < 1$. Based on these conditions, check whether the following AR(2) processes are covariance stationary. Answer Yes or No and in case your answer is No, mention at least one condition that is violated.

   1. $\phi_1 = 0.5$ and $\phi_2 = 0.4$,
   2. $\phi_1 = -1.9$ and $\phi_2 = -0.9$,
   3. $\phi_1 = 1$ and $\phi_2 = 0$.

7. Derive explicit expressions for the one- and two-step ahead forecasts, i.e., $\mathbb{E}_t[X_{t+1}] \equiv X_{t+1|t}$ and $\mathbb{E}_t[X_{t+2}] \equiv X_{t+2|t}$, of the AR(2) process given in Question 4. In both cases, provide the forecasting error and the variance of the forecasting error.
Hint: Find expressions for the forecasting errors in terms of ϵ_{t+1} and ϵ_{t+2}, respectively.

8. Derive the best linear projection of X_t on the space engendered by $(\mathbf{1}, X_{t-1}, X_{t-2})$, where $\mathbf{1}$ is a constant random variable. Denote the corresponding coefficients by a_0, a_1, and a_2.
Hint: Write down the orthogonality conditions and the corresponding inner products. Then use the orthogonality condition with respect to $\mathbf{1}$ to write all variables in mean-deviation in the other equations. Finally, solve for a_1 and a_2.

9. Using the results of the previous question, propose an explicit estimator for μ, ϕ_1, and ϕ_2.

Exercise 2: Questions related to the course

Precise answers are gratefully acknowledged.

1. What are the main stylized facts regarding daily volatility as captured by squared returns and absolute returns? Explain carefully.

2. An independent and identically distributed stochastic process $(X_t, t \in \mathbb{Z})$ is always weakly stationary? TRUE or FALSE? Explain carefully.

3. Consider a portfolio consisting of two assets with returns $r_{1,t}$ and $r_{2,t}$, with the following moments:

$$\begin{aligned} \mu_1 &= \mathbb{E}\left[r_{1,t}\right], \mu_2 = \mathbb{E}\left[r_{2,t}\right] \\ \sigma_1^2 &= \mathbb{E}\left[(r_{1,t}-\mu_1)^2\right], \sigma_2^2 = \mathbb{E}\left[(r_{2,t}-\mu_2)^2\right] \\ \sigma_{1,2} &= \mathbb{E}\left[(r_{1,t}-\mu_1)(r_{2,t}-\mu_2)\right]. \end{aligned}$$

The vector of portfolio weights is such that $w_1 + w_2 = 1$, where w_1 (respectively, w_2) is the relative contribution of the first asset (respectively, second asset) in the portfolio. Notably, the optimal weights of the global minimum variance portfolio are given by:

$$w^\star = \frac{1}{e_2^\top \Sigma^{-1} e_2} \Sigma^{-1} e_2$$

where $w^\star = (w_1^\star, w_2^\star)^\top$, $e_2 = (1,1)^\top$ and Σ is the variance-covariance matrix of $(r_{1,t}, r_{2,t})^\top$. Consider now the following linear regression model in the population:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

with

$$\begin{aligned} y_t &= r_{2,t} \\ x_t &= r_{2,t} - r_{1,t}, \end{aligned}$$

where u_t denotes the error term, with $\mathbb{E}\left[u_t\right] = 0$ and $\mathbb{V}\left[u_t\right] = \sigma^2$, that satisfies the classical assumptions of the simple linear regression model.

- Derive the population orthogonality conditions.
- Solve these orthogonality conditions for β_1 and show that:

$$\beta_1 = \frac{\sigma_2^2 - \sigma_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}}.$$

- How does it compare with $w_1^\star$?

4. Consider a given (weakly stationary) time series with $T = 200$ observations. One estimates several ARMA(p,q) specifications and reports some selected information criteria (the Akaike information criterion, AIC, and the Schwartz Bayesian information criterion, SBIC). These two information criteria have not been corrected for small sample size issues. Results are provided in Table 1.

Table 1: Information criteria

(p,q)	AIC	SBIC	(p,q)	AIC	SBIC
(6,12)	0.963	1.173	(6,11)	0.941	1.141
(6,10)	0.938	1.126	(5,11)	0.950	1.138
(5,10)	0.985	1.162	(5,9)	0.939	1.082
(5,6)	0.986	1.119	(5,5)	0.976	1.087
(5,3)	1.088	1.187	(5,2)	1.276	1.376

- What is an information criterion? Explain carefully.
- What are the main differences between these two information criteria (in terms of statistical properties)?
- Which model(s) can one select? Explain carefully.

5. The CIR model, also known as the square-root process, can be expressed as:

$$\begin{aligned} dY(t) &= (\alpha - \beta Y(t))\, dt + \sigma\sqrt{Y(t)} dW(t) \\ Y(0) &= y_0 \end{aligned}$$

where $t \in [0, T]$, Y(t) is, for example, the short-term interest rate, $y_0 > 0$ is given, $W(t)$ is a standard Wiener process, and the coefficients satisfy all standard regularity conditions. Using the transformation $X(t) = \ln(Y(t))$, one gets for $t \in [0, T]$ (using Itô's lemma):

$$\begin{aligned} dX(t) &= \left(\frac{\alpha}{\exp(X(t))} - \beta - \frac{\sigma^2}{2\exp(X(t))} \right) dt + \frac{\sigma}{\sqrt{\exp(X(t))}} dW(t) \\ X(0) &= x_0. \end{aligned}$$

5.1. Why might this transformation be useful? Explain carefully.

5.2. Suppose that the time increment is constant and thus the equally-spaced observations are available at $(t_i, i = 0, \cdots, n)$. After defining the time increment as well as all of your notations (e.g., t_i), provide the Euler-based approximation of $(X(t), t \in [0, T])$.

5.3. Provide the Milstein-based approximation of $(X(t), t \in [0, T])$.

5.4. Using the result of Question 5.2., provide the conditional distribution of X_{t_i} given $X_{t_{i-1}} = x_{t_{i-1}}$ for $i \geq 1$.

5.5. Write down the corresponding conditional log-likelihood function.

6. Consider the following 1-factor model for two monthly assets, asset 1 and asset 2, with returns denoted r_1 and r_2:

$$
\begin{aligned}
r_{1,t} &= \beta_1 r_{m,t} + \epsilon_{1,t} \\
r_{2,t} &= \beta_2 r_{m,t} + \epsilon_{2,t}
\end{aligned}
$$

where r_m is the monthly return on the S&P500, $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are independent of each other and i.i.d. It is further assumed that they are uncorrelated with $r_{m,t}$.
In addition,

- The variance of ϵ_1 and ϵ_2 are given by $\sigma_1^2 = 0.07$ and $\sigma_2^2 = 0.04$;
- The mean of the monthly S&P500 return is estimated to 0.007;
- The volatility of the monthly S&P500 return is estimated by using a GARCH(1,1) model;
- The betas for the two stocks are $\beta_1 = 0.8$ and $\beta_2 = 1.1$.

6.1. What is the expected monthly return on assets 1 and 2?

6.2. In the sequel, treat the mean of r_1 and r_2 as zero. Suppose that the monthly volatility of the S&P500 return is $\sqrt{0.030}$. Find the variances of asset 1 and asset 2, and the covariance between the two assets.

6.3. Find the variance of a portfolio that allocates 0.5 of the wealth in asset 1 and 0.5 of the wealth in asset 2 when the market volatility is $\sqrt{0.030}$.

6.4. In the case of an N-asset portfolio, why is this one-factor model useful? Explain carefully.

6.5. How would you implement a resampling procedure for the mean-variance frontier using the one-factor model?

Exercise 3: Interpretation

Short and precise answers are gratefully acknowledged.

1. Consider the following three-equation model, that describes the dynamics of the first-difference of the (log of) USD/EUR exchange rate:

$$\begin{aligned} \Delta s_t &= c + u_t \\ u_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

where $Z_t \sim \texttt{i.i.d.}\mathcal{N}(0,1)$ and s_t is the logarithm of the USD/EUR exchange rate.

1.1. Interpret σ_t^2.

1.2. Describe the motivation behind this model and the features of the log return of the USD/EUR exchange rate it aims to capture.

1.3. This model is estimated using monthly data for the period 1999M2-2012M12 and the estimation results are provided in Table 2.

Table 2: Estimation results

	Coefficient	Std. Error	t-stat	p-value
c	0.0728	0.1093	0.6655	0.5057
ω	0.00281	0.00258	1.0895	0.2759
α	0.11725	0.05977	1.9616	0.0498
β	0.73018	0.16423	4.44617	0.0000

(i) Is there evidence that buyers of the Euro have been making a positive return over the sample period? Why?

(ii) What is the interpretation of the estimates of α and β?

(iii) What is the long-run average (unconditional) variance? What is the interpretation?

2. Suppose that you have two monthly times series (over the period 1975M1 - 2010M12) at hand: (1) Average home sales price and (2) Average home rental price.

2.1. Explain why these two prices might be cointegrated. You should consider economic/financial arguments and explain the concept of cointegration.

2.2. How would you test the presence of a cointegrating relationship assuming you know the cointegrating vector? Explain carefully.

2.3. Suppose that you proceed with the estimation of an error correction model (ECM). Using these estimates, the representation of the error correction model is given by:

$$\begin{pmatrix} \Delta s_t \\ \Delta r_t \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.40 & 0.10 \\ 0.07 & 0.50 \end{pmatrix} \begin{pmatrix} \Delta s_{t-1} \\ \Delta r_{t-1} \end{pmatrix} + \begin{pmatrix} -0.15 \\ 0.10 \end{pmatrix} z_{t-1} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

where $z_t = s_t - r_t$, s_t is the average home sale price, and r_t is the average home rental price. All coefficients are assumed to be statistically different from zero.

- Provide an interpretation of the coefficients on z_{t-1}. Do they make sense? Explain carefully.

- Suppose that $z_{t-1} > 0$, What would happen?

2.4. Suppose now that both prices follow a random walk, but you conclude that the two series are not cointegrated. Explain carefully how you would model the two series in this case. Be explicit as much as you can.

3. Let (X_t) denote a financial time series (Figure 1) in the US over the period 1960Q1-2010Q1 (quarterly data). The autocorrelation function and the partial autocorrelation functions are provided in Figure 2 and Figure 3. Using the information provided by Figure 2 and Figure 3, one estimates two models: MA(4) and AR(2). The corresponding estimates are given in Table 3. Finally the autocorrelation functions of the residuals are reported in Figure 4 (MA(4)) and in Figure 5 (AR(2)). All figures are reported in Annex 1.

3.1. Comment Table 3.

3.2. Using Table 1 and Figures 1-5, which specification is preferable? Explain carefully (with different arguments).

3.3. What are the main drawbacks (limits) of your answer in the previous question? Explain carefully.

Table 3: ARMA estimation

	AR(2)	MA(4)
Constant	5.797***	5.835***
	(9.99)	(10.33)
ϕ_1	1.596***	
	(38.65)	
ϕ_2	-0.636***	
	(-9.26)	
θ_1		1.595***
		(26.23)
θ_2		1.701***
		(25.26)
θ_3		1.489***
		(20.86)
θ_4		0.677***
		(10.36)

Note: t-statistics are reported in parentheses. By convention, $*$ means that the p-value is less than 0.05, $**$ (respectively, $***$) means that the p-value is less than 0.01 (respectively, 0.001).

4. Consider the US 3-month treasury bill rate over the period 1960m1-1995m4 (Figure 6). The (partial) autocorrelation function of this variable and its first-difference are provided in Figure 7 and in Figure 8. All figures are reported in Annex 2.

4.1. Comment Figures 6, 7, and 8.

4.2. In order to assess whether the series might be stationary or non-stationary, one conducts unit root tests. As a first step, a standard Dickey-Fuller test is implemented. The dependent variable is the first-difference of the 3-month treasury bill rate. Results are provided in Table 4.

Table 4: Results of the DF test

Parameter	Estimate	Std. Error	t-stat.	p-value
	A. Spectral OLS autoregression			
ϕ	-0.0197	0.0095	-2.0712	0.0389
c	0.1242	0.0645	1.9248	0.0549

- Comment the specification of the DF test.
- Write down the test regression.
- Write down the null and alternative hypothesis.
- Taking that the critical values are -3.445 (1% level), -2.868 (5% level), and -2.570 (10% level), what can be inferred from this test?

4.3. In a second step, an Augmented Dickey-Fuller (ADF) test is implemented with the first-difference of the 3-month treasury bill rate as dependent variable. Using some information criteria, a specification with 6 lags is considered. Results are provided in Table 5.

Table 5: Results of the ADF test

Parameter	Estimate	Std. Error	t-stat.	p-value
ϕ	-0.0161	0.0090	-1.7902	0.0741
c	0.1022	0.0605	1.6896	0.0919
α_1	0.3508	0.0472	7.4312	0.0000
α_2	-0.2155	0.0497	-4.3291	0.0000
α_3	0.0255	0.0506	0.5057	0.6134
α_4	-0.1033	0.0504	-2.0469	0.0413
α_5	0.1537	0.0494	3.1055	0.0020
α_6	-0.2555	0.0472	-5.4038	0.0000

- Why might it be preferable to use an augmented Dickey-Fuller test rather than a standard Dickey-Fuller test? Explain carefully.
- Write down the value of the augmented Dickey-Fuller student-based statistic.
- Taking that the critical values are -3.445 (1% level), -2.868 (5% level), and -2.570 (10% level), what can be inferred from this test?

4.4. As a final step, a PP (Phillips-Perron) test is also conducted. The test statistic is given by $Z_t = -1.8276$.

* Write down the test regression.
* Taking that the critical values are -3.445 (1% level), -2.868 (5% level), and -2.570 (10% level), what can be inferred from this test?

4.5. All in all, using all of the previous results, which order of integration might be suggested for the 3-month treasury bill rate?

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Annex 1: Financial series (Exercise 3, Question 3)

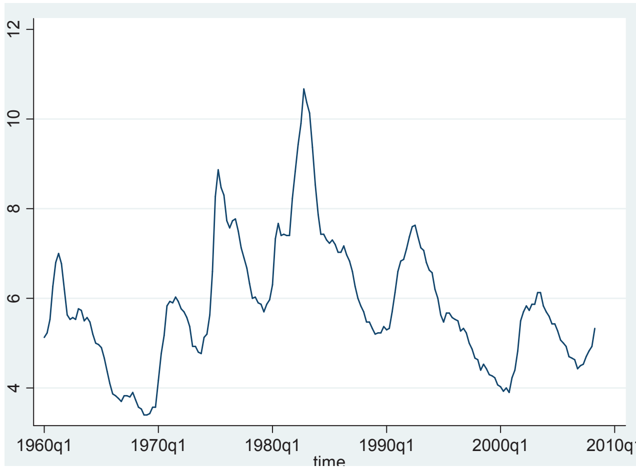


Figure 1: Financial time series

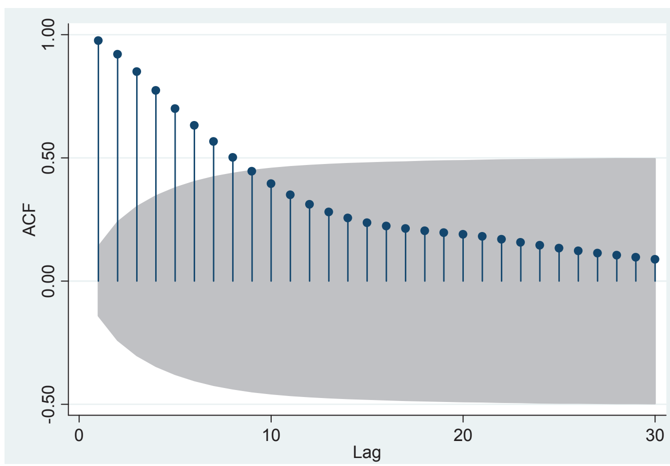


Figure 2: Autocorrelation function

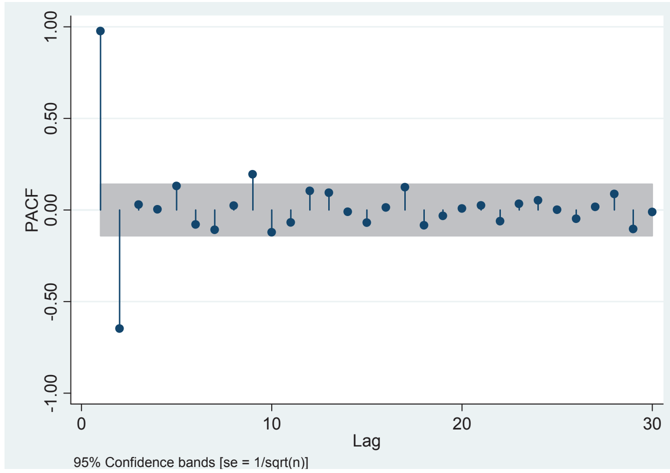


Figure 3: Partial autocorrelation function

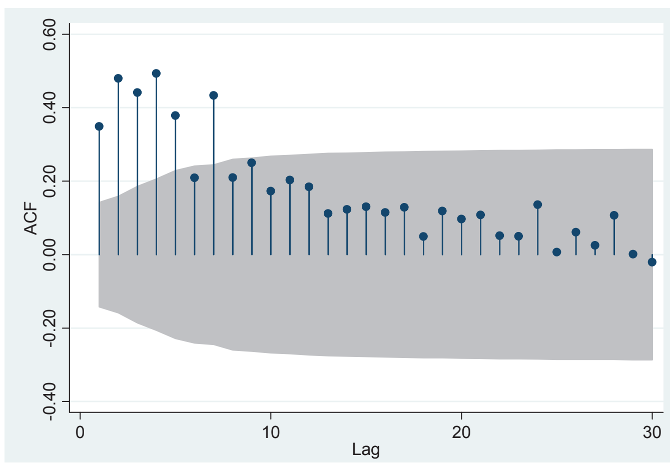


Figure 4: Autocorrelation function of the residuals using a MA(4)

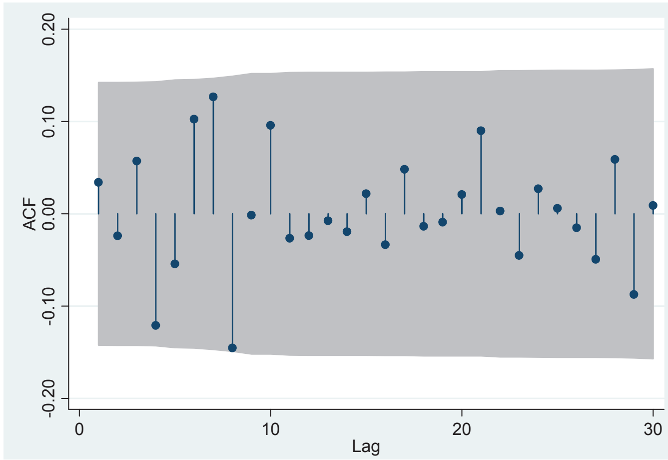


Figure 5: Autocorrelation function of the residuals using an AR(2)

Annex 2: Three-month treasury Bill rate (Exercise 3, Question 4)

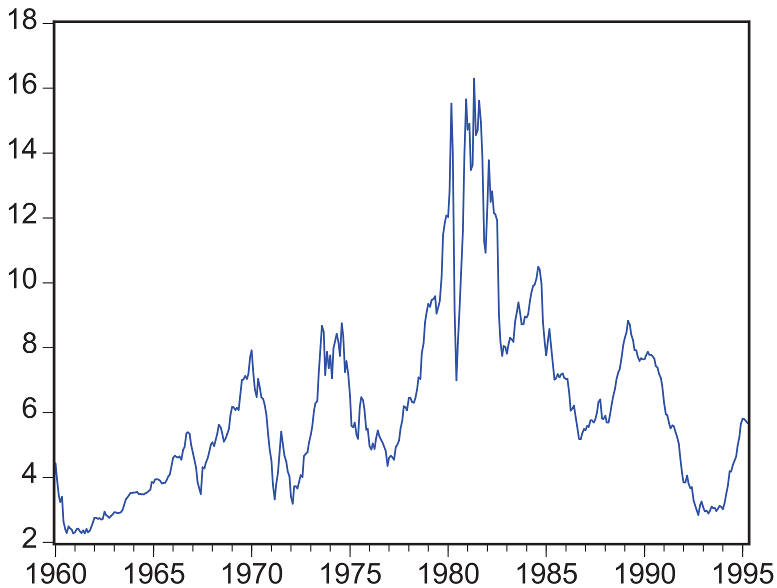
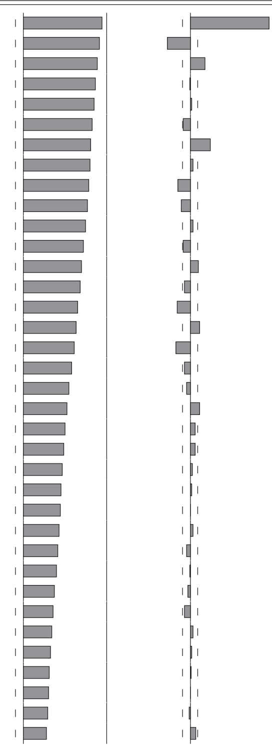


Figure 6: Three-month treasury bill rate



Lag	AC	PAC	Q-Stat	Prob
1	0.980	0.980	410.16	0.000
2	0.949	-0.284	795.88	0.000
3	0.923	0.182	1161.3	0.000
4	0.900	-0.001	1509.9	0.000
5	0.879	0.020	1843.3	0.000
6	0.856	-0.087	2159.9	0.000
7	0.839	0.246	2465.1	0.000
8	0.830	0.035	2764.4	0.000
9	0.817	-0.153	3055.0	0.000
10	0.796	-0.110	3331.3	0.000
11	0.772	0.038	3591.9	0.000
12	0.748	-0.085	3837.3	0.000
13	0.728	0.104	4070.4	0.000
14	0.707	-0.073	4290.9	0.000
15	0.680	-0.159	4494.9	0.000
16	0.656	0.119	4685.4	0.000
17	0.632	-0.174	4862.5	0.000
18	0.604	-0.068	5024.7	0.000
19	0.572	-0.045	5170.6	0.000
20	0.541	0.119	5301.5	0.000
21	0.519	0.059	5422.4	0.000
22	0.504	0.058	5536.5	0.000
23	0.489	0.027	5644.1	0.000
24	0.474	0.019	5745.5	0.000
25	0.461	0.007	5841.8	0.000
26	0.449	0.038	5933.1	0.000
27	0.434	-0.045	6018.8	0.000
28	0.414	-0.002	6096.8	0.000
29	0.392	-0.027	6167.0	0.000
30	0.372	-0.070	6230.6	0.000
31	0.355	0.039	6288.5	0.000
32	0.340	0.022	6341.6	0.000
33	0.325	0.009	6390.4	0.000
34	0.313	0.005	6435.9	0.000
35	0.303	-0.015	6478.7	0.000
36	0.295	0.071	6519.2	0.000

Figure 7: (Partial) Autocorrelation function of the three-month treasury bill rate

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.273	0.273	31.714	0.000
		2	-0.117	-0.206	37.526	0.000
		3	-0.103	-0.009	42.059	0.000
		4	-0.042	-0.033	42.822	0.000
		5	0.055	0.064	44.129	0.000
		6	-0.181	-0.260	58.211	0.000
		7	-0.192	-0.048	74.148	0.000
		8	0.103	0.148	78.766	0.000
		9	0.208	0.093	97.602	0.000
		10	0.068	-0.045	99.597	0.000
		11	-0.006	0.070	99.614	0.000
		12	-0.106	-0.122	104.55	0.000
		13	0.034	0.065	105.05	0.000
		14	0.171	0.151	117.97	0.000
		15	-0.102	-0.140	122.55	0.000
		16	0.009	0.171	122.59	0.000
		17	0.098	0.045	126.84	0.000
		18	0.098	0.023	131.11	0.000
		19	-0.021	-0.133	131.30	0.000
		20	-0.240	-0.079	156.96	0.000
		21	-0.161	-0.071	168.64	0.000
		22	-0.003	-0.044	168.65	0.000
		23	-0.007	-0.024	168.67	0.000
		24	-0.048	-0.026	169.73	0.000
		25	-0.015	-0.052	169.83	0.000
		26	0.066	0.033	171.81	0.000
		27	0.133	-0.013	179.85	0.000
		28	0.048	0.015	180.89	0.000
		29	-0.063	0.065	182.69	0.000
		30	-0.051	-0.058	183.87	0.000
		31	-0.054	-0.033	185.23	0.000
		32	-0.028	-0.021	185.58	0.000
		33	-0.062	-0.020	187.33	0.000
		34	-0.049	0.012	188.44	0.000
		35	-0.038	-0.090	189.12	0.000
		36	-0.030	-0.021	189.53	0.000

Figure 8: (Partial) Autocorrelation function of the first-difference of three-month treasury bill rate