Fundamentals of Traffic Operations and Control
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Exercise solutions
Macroscopic fundamental diagram

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a) We can visualize the dynamics of the city center as a single tank with water level representing the accumulation n, together with entering flows q_1 and q_2, and exiting flow $G(n)$ (representing the network outflow as modelled by the macroscopic fundamental diagram (MFD)), as depicted in fig. 1. We can write the corresponding mass conservation equation as:

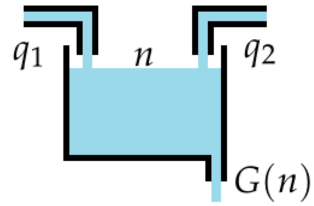

Figure 1: Water tank representing city center dynamics.

$$\frac{dn}{dt} = q_1 + q_2 - G(n). \tag{1}$$

b) The given MFD is a piecewise affine function with three pieces (we call them $G_1(n)$, $G_2(n)$, $G_3(n)$, as in fig. 2) as follows:

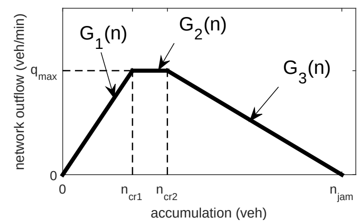

Figure 2: Macroscopic fundamental diagram.

$$G_1(n) = \frac{q_{\text{max}}}{n_{cr1}} \cdot n = \frac{100 \text{ veh/min}}{1000 \text{ veh}} \cdot n = 0.1 \text{ min}^{-1} \cdot n$$
$$G_2(n) = q_{\text{max}} = 100 \text{ veh/min}$$
$$G_3(n) = \frac{q_{\text{max}} \cdot (n_{\text{jam}} - n)}{n_{\text{jam}} - n_{cr2}} = 0.04 \text{ min}^{-1} \cdot (4000 \text{ veh} - n).$$

From 7am until n reaches n_{cr1}, the piece $G_1(n)$ is active, that is:

$$\frac{dn}{dt} = q_1 + q_2 - G_1(n) = 150 \text{ veh/min} - 0.1 \text{ min}^{-1} \cdot n.$$

We call the time n reaches n_{cr1} as t_1 and calculate it as follows:

$$\begin{aligned}
&\frac{dn}{dt} = 150 - 0.1 \cdot n \\
&\frac{dn}{150 - 0.1 \cdot n} = dt \\
&\int_{n(0)}^{n_{cr1}} \frac{dn}{150 - 0.1 \cdot n} = \int_0^{t_1} dt \\
&- 10 \cdot \ln(1500 - n) \,\Big|_{500}^{1000} = t \,\Big|_0^{t_1} \\
&- 10 \cdot \ln(500) + 10 \cdot \ln(1000) = t_1 \\
&\implies t_1 = 6.9315 \text{ min} \approx 7 \text{ min.}
\end{aligned}$$

From 7:07am until n reaches n_{cr2}, the piece $G_2(n)$ is active, that is:

$$\frac{dn}{dt} = q_1 + q_2 - G_2(n) = 50 \text{ veh/min}.$$

We call the time n reaches n_{cr2} as t_2 and calculate it as follows:

$$\begin{aligned}
&\frac{dn}{dt} = 50 \\
&\frac{dn}{50} = dt \\
&\int_{n_{cr1}}^{n_{cr2}} \frac{dn}{50} = \int_{t_1}^{t_2} dt \\
&\frac{1}{50} \cdot n \,\Big|_{1000}^{1500} = t \,\Big|_{7}^{t_2} \\
&\frac{1}{50} \cdot (1500 - 1000) = t_2 - 7 \\
&\implies t_2 = 17 \text{ min.}
\end{aligned}$$

From 7:17am on, the piece $G_3(n)$ is active, that is:

$$\frac{dn}{dt} = q_1 + q_2 - G_3(n) = 150 \text{ veh/min} - 0.04 \text{ min}^{-1} \cdot (4000 \text{ veh} - n).$$

We call the accumulation at 7:30am as n^* and calculate it as follows:

$$\begin{aligned}
&\frac{dn}{dt} = 150 - 0.04 \cdot (4000 - n) \\
&\frac{dn}{dt} = -10 + 0.04 \cdot n \\
&\frac{dn}{-10 + 0.04 \cdot n} = dt \\
&\int_{n_{cr2}}^{n^*} \frac{dn}{-10 + 0.04 \cdot n} = \int_{t_2}^{30} dt \\
&25 \cdot \ln(n - 250) \,\Big|_{n_{cr2}}^{n^*} = t \,\Big|_{t_2}^{30} \\
&25 \cdot (\ln(n^* - 250) - \ln(1250)) = 13 \\
&\implies n^* = n(t = 30 \text{ min}) = 2352.5 \text{ veh.}
\end{aligned}$$

c) With perimeter control, the dynamics contains a restriction term u, as depicted in fig. 3. The dynamical equation becomes:

$$\frac{dn}{dt} = u \cdot q_1 + q_2 - G(n).$$

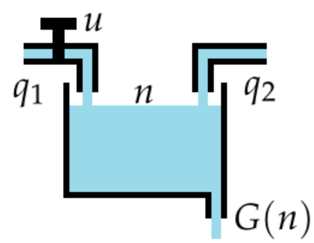

Figure 3: Water tank representation with perimeter control.

We call the constant perimeter control value that would result in an accumulation of 2500 vehicles at 8:00am as u^* and calculate it as follows (noting that the piece $G_3(n)$ will be active since $n > n_{cr2}$ from 7:30am to 8:00am):

$$
\begin{aligned}
&\frac{dn}{dt} = u^* \cdot q_1 + q_2 - G_3(n) \\
&\frac{dn}{dt} = u^* \cdot 90 - 100 + 0.04 \cdot n \\
&\frac{dn}{u^* \cdot 90 - 100 + 0.04 \cdot n} = dt \\
&\int_{2352.5}^{2500} \frac{dn}{u^* \cdot 90 - 100 + 0.04 \cdot n} = \int_{30}^{60} dt \\
&25 \cdot \ln(u^* \cdot 90 - 100 + 0.04 \cdot n) \Big|_{2352.5}^{2500} = t \Big|_{30}^{60} \\
&25 \cdot (\ln(u^* \cdot 90 - 100 + 0.04 \cdot 2500) - \ln(u^* \cdot 90 - 100 + 0.04 \cdot 2352.5)) = 30 \\
&\implies u^* \approx 0.094.
\end{aligned}
$$

d) We can model the queue outside the city center as an accumulation m, with entering flow q_1 and exiting flow $u \cdot q_1$, and write its dynamics as follows:

$$\frac{dm}{dt} = q_1 - u^* \cdot q_1 = 90 - 0.094 \cdot 90 \approx 81.5 \text{ veh/min}.$$

We call the queue accumulation at 8:00am as m^* and calculate it as follows (noting that the queue will be empty at 7:30am and will grow once the perimeter control is activated at 7:30am):

$$
\begin{aligned}
&\frac{dm}{dt} = 81.5 \\
&\frac{dm}{81.5} = dt \\
&\int_0^{m^*} \frac{dm}{81.5} = \int_{30}^{60} dt \\
&\frac{1}{81.5} \cdot m \Big|_0^{m^*} = t \Big|_{30}^{60} \\
&\frac{1}{81.5} \cdot m^* = 30 \\
&\implies m^* = 2445 \text{ veh.}
\end{aligned}
$$

e) We can get numerical solutions (see fig. 4) by discretizing eq. (1) in time via, for example, the forward Euler method, and then using it inside a for loop:

```
n(1) = 500; % initialize accumulation
T = 0.1; % time step (min)
tmax = 30; % final time (min)
kmax = tmax/deltaT; % number of iterations
for k = 1:kmax
    % discrete-time version of equation 1
    n(k+1) = n(k) + T*(150 - G(n(k)));
end
```

where G is a function expressing the MFD.

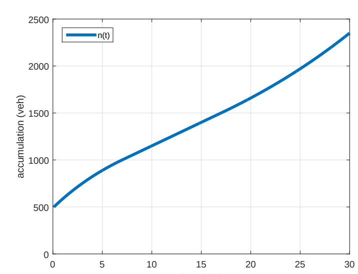

Figure 4: Trajectory of accumulation n from 7:00am to 7:30am.