Fundamentals of Traffic Operations and Control
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Exercise solutions
Shockwave theory

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a) We make use of the following formulas for the average cycle link flow and density:

$$\text{average flow: } Q = \frac{VKT}{S_{\text{total}}} = \frac{\sum_i S_i \cdot q_i}{L \cdot T_\Sigma}$$
$$\text{average density: } K = \frac{VHT}{S_{\text{total}}} = \frac{\sum_i S_i \cdot k_i}{L \cdot T_\Sigma},$$

where i is the state of traffic vehicles experience when traveling through the road and traffic signal, S_i is the area of state i in the time-space diagram, q_i and k_i are the flow and density corresponding to state i, S_{total} is the total area of the cycle on the time-space diagram, whereas T_Σ is the cycle duration (i.e., sum of green time T_G and red time T_R), with $T_\Sigma = T_G + T_R = 30 \text{ s} + 30 \text{ s} = 60 \text{ s}$. The traffic states for the system in question are: A (vehicles approaching the traffic signal), J (vehicles stopping at the red light), C (vehicles discharging from the queue).

We can draw the fundamental diagram (FD) (using the values given in the question) as shown in fig. 1, with points A, J, and C corresponding to the three traffic states, while v_f is the free flow speed that can be calculated from the FD as follows:

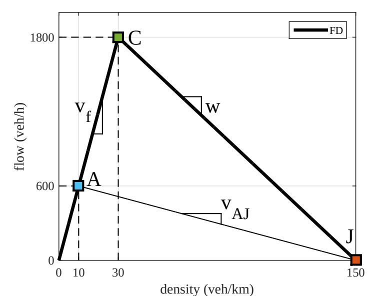

Figure 1: Fundamental diagram.

$$v_f = \frac{|q_C|}{|k_C|} = \frac{1800 \text{ veh/h}}{30 \text{ veh/km}} = 60 \text{ km/h},$$

whereas w is the discharge wave speed that can again be calculated from the FD as follows:

$$w = \frac{|q_C - q_J|}{|k_C - k_J|} = \frac{|1800 \text{ veh/h} - 0 \text{ veh/h}|}{|30 \text{ veh/km} - 150 \text{ veh/km}|} = 15 \text{ km/h}.$$

As the approaching vehicles join the queue at the traffic light, the traffic state will transition from point A to point J, with shockwave speed v_{AJ}, which is the slope of the line joining points A and J:

$$v_{AJ} = \frac{|q_A - q_J|}{|k_A - k_J|} = \frac{|600 \text{ veh/h} - 0 \text{ veh/h}|}{|10 \text{ veh/km} - 150 \text{ veh/km}|} \approx 4.286 \text{ km/h}.$$

To calculate the areas corresponding to the traffic states, we draw the time-space diagram as given in fig. 2 (note that only part of the figure with the queue is shown here, as the blue area extends downwards since $L = 300$ m).

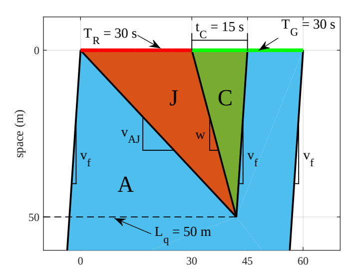

Figure 2: Time-space diagram.

The area of state J is the area of the red triangle in fig. 2:

$$S_J = \frac{1}{2} \cdot T_R \cdot L_q,$$

where L_q is the maximum queue length that can be obtained from the time-space diagram as follows:

$$L_q = \frac{w \cdot v_{AJ}}{w - v_{AJ}} \cdot T_R = \frac{15 \text{ km/h} \cdot 4.286 \text{ km/h}}{15 \text{ km/h} - 4.286 \text{ km/h}} \cdot 30 \text{ s} = 50 \text{ m},$$

thus we obtain S_J as follows:

$$S_J = \frac{1}{2} \cdot 30 \text{ s} \cdot 50 \text{ m} = 750 \text{ m.s.}$$

The area of state C is the area of the green triangle in fig. 2:

$$S_C = \frac{1}{2} \cdot t_C \cdot L_q,$$

where t_C can be obtained from the time-space diagram as follows:

$$t_C = \frac{L_q}{w} + \frac{L_q}{v_f} = \frac{50 \text{ m}}{15 \text{ km/h}} + \frac{50 \text{ m}}{60 \text{ km/h}} = 15 \text{ s},$$

thus we obtain S_C as follows:

$$S_C = \frac{1}{2} \cdot 15 \text{ s} \cdot 50 \text{ m} = 375 \text{ m.s.}$$

The total area for the whole cycle S_{total} is equal to $L \cdot T_\Sigma$:

$$S_{\text{total}} = L \cdot T_\Sigma = 300 \text{ m} \cdot 60 \text{ s} = 18000 \text{ m.s},$$

which is also the sum of the areas of the three traffic states:

$$S_{\text{total}} = S_A + S_J + S_C,$$

thus we can find S_A as follows:

$$S_A = S_{\text{total}} - (S_J + S_C) = 18000 \text{ m.s} - (750 + 375) \text{ m.s} = 16875 \text{ m.s.}$$

Using the formula for Q we obtain:

$$\begin{aligned} Q &= \frac{q_A \cdot S_A + q_J \cdot S_J + q_C \cdot S_C}{L \cdot T_\Sigma} \\ &= \frac{600 \text{ veh/h} \cdot 16875 \text{ m.s} + 0 \text{ veh/h} \cdot 750 \text{ m.s} + 1800 \text{ veh/h} \cdot 375 \text{ m.s}}{18000 \text{ m.s}} \\ &= 600 \text{ veh/h}, \end{aligned}$$

and the formula for K, we obtain:

$$\begin{aligned} K &= \frac{k_A \cdot S_A + k_J \cdot S_J + k_C \cdot S_C}{L \cdot T_\Sigma} \\ &= \frac{10 \text{ veh/km} \cdot 16875 \text{ m.s} + 150 \text{ veh/km} \cdot 750 \text{ m.s} + 30 \text{ veh/km} \cdot 375 \text{ m.s}}{18000 \text{ m.s}} \\ &= 16.25 \text{ veh/km}. \end{aligned}$$

b) We can consider a time-space diagram describing the situation in the question as shown in fig. 3. Here h is the headway, denoting the time it takes for the overpassing vehicle to pass two consecutive vehicles in the stream, arbitrarily named a and b in the figure, while s is the spacing of the stream.

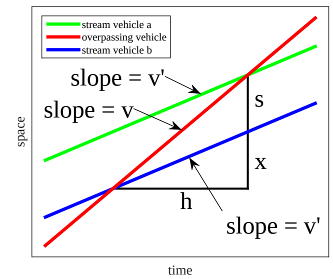

Figure 3: Time-space diagram.

From the figure, we can see that:

$$v = \frac{s+x}{h}$$

and

$$v' = \frac{x}{h}.$$

Combining the two, we get:

$$s = (v - v') \cdot h.$$

Remembering that the density k' is the inverse of spacing s, that is:

$$k' = \frac{1}{s}$$

we can write:

$$s = \frac{1}{k'} = (v - v') \cdot h$$

which is the same as:

$$\frac{1}{h} = (v - v') \cdot k'.$$

Remembering that the flow q' (i.e., the passing rate) is the inverse of headway h, that is:

$$q' = \frac{1}{h}$$

we can find the passing rate as follows:

$$q' = (v - v') \cdot k'.$$