QUANTUM PHYSICS III

Solutions to Problem Set 3

6 October 2017

1. Completeness of Coherent states

1. To compute the integral
$$\int d^2\alpha e^{-|\alpha|^2}\alpha^{*n}\alpha^m \ , \tag{1}$$
it is convenient to use polar coordinates,
$$\mathrm{Re}\,\alpha = r\cos\phi \ , \quad \mathrm{Im}\,\alpha = r\sin\phi \ . \tag{2}$$
The integral becomes
$$\begin{aligned}\int d\phi dr\, re^{-r^2}r^{n+m}e^{i(m-n)\phi} &= \int_0^\infty dre^{-r^2}r^{n+m+1}\int_0^{2\pi} d\phi e^{i(m-n)\phi} \\ &= 2\pi\delta_{nm}\int_0^\infty drr^{2n+1}e^{-r^2} = \pi\delta_{nm}n! \ .\end{aligned} \tag{3}$$

2. Recall that the coherent state $|\alpha\rangle$ is related to the eigenstates of the harmonic oscillator via
$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle \ . \tag{4}$$
Hence, we can write
$$\begin{aligned}\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| &= \int \frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| \\ = \sum_{n,m}\frac{|n\rangle\langle m|}{\sqrt{n!}\sqrt{m!}}\frac{1}{\pi}\int d^2\alpha e^{-|\alpha|^2}\alpha^n\alpha^{*m} &= \sum_{n,m}\frac{|n\rangle\langle m|}{\sqrt{n!}\sqrt{m!}}n!\delta_{nm} = \sum_n |n\rangle\langle n| = 1 \ .\end{aligned} \tag{5}$$

3. Using eq. (4), we have
$$\begin{aligned}\langle\alpha|\alpha'\rangle &= e^{-\frac{|\alpha|^2}{2}-\frac{|\alpha'|^2}{2}}\sum_{n,m=0}^\infty \frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha'^{m}}{\sqrt{m!}}\langle n|m\rangle \\ &= e^{-\frac{|\alpha|^2}{2}-\frac{|\alpha'|^2}{2}}\sum_{n=0}^\infty \frac{(\alpha^*\alpha')^n}{n!} = e^{-\frac{1}{2}|\alpha-\alpha'|^2}e^{i\,\mathrm{Im}(\alpha'\alpha^*)} \ .\end{aligned} \tag{6}$$
Thus, the coherent states are not orthogonal to each other. We see, however, that they become approximately orthogonal in the limit of large separation of states, that is when $|\alpha - \alpha'| \to \infty$.

4. By the means of eqs. (5) and (6), we get

$$|\alpha\rangle = \frac{1}{\pi}\int d^2\alpha'|\alpha'\rangle\langle\alpha'|\alpha\rangle = \frac{1}{\pi}\int d^2\alpha' e^{-\frac{1}{2}|\alpha-\alpha'|^2} e^{i\,\mathrm{Im}(\alpha'\alpha^*)}|\alpha'\rangle\ . \tag{7}$$

Hence, any coherent state can be expressed through other coherent states. This proves the system of coherent states to be overcomplete.

5. Let us again make use of eq. (4) and write

$$\int d^2\alpha f(|\alpha|)|\alpha\rangle = \sum_n \left(\int d^2\alpha\ f(|\alpha|)e^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle = 0\ . \tag{8}$$

Since the eigenstates $\{|n\rangle\ ,\ n = 0, 1, 2, ...\}$ of the harmonic oscillator do form a basis in the Hilbert space, from the above it follows that

$$\int d^2\alpha\ f(|\alpha|)e^{-\frac{|\alpha|^2}{2}}\alpha^n = 0\ ,\quad \text{for all } n\,. \tag{9}$$

In polar coordinates, the last integral is rewritten as

$$\begin{aligned}\int d\phi dr\ rf(r)e^{-\frac{r^2}{2}}r^n e^{in\phi} &= \int_0^\infty dr\ rf(r)r^n e^{-\frac{r^2}{2}}\cdot\int_0^{2\pi} d\phi\ e^{in\phi}\\ &= \delta_{n0}2\pi\int_0^\infty dr\ rf(r)e^{-\frac{r^2}{2}}\ .\end{aligned} \tag{10}$$

To ensure eq. (8), it is enough to choose $f(r) = \dfrac{e^{\frac{r^2}{2}}}{r}\cdot g(r)$, where $g(r)$ is such that

$$\int_0^\infty dr\ g(r) = 0\ . \tag{11}$$

For example,

$$g(r) = 2e^{-2r} - e^{-r}\ . \tag{12}$$

2. The Saddle-point method

1. Let x_0 be a single finite extremum point of the function $f(x)$. Expanding around x_0 up to the second order, we have

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + O((x-x_0)^3)\ . \tag{13}$$

Here we assume that the second derivative $f''(x)$ is not vanishing at $x = x_0$. Substituting this into the integral

$$I = \int_{-\infty}^{\infty} dx\ e^{\frac{i}{\hbar}f(x)-\epsilon|x|} \tag{14}$$

and neglecting ϵ, we obtain [1]

$$I \approx e^{\frac{i}{\hbar}f(x_0)} \int_{-\infty}^{\infty} dy\, e^{\frac{i}{\hbar}\frac{f''(x_0)}{2}y^2} = e^{\frac{i}{\hbar}f(x_0)} \sqrt{\frac{2\pi\hbar i}{f''(x_0)}} \ . \tag{15}$$

We conclude that the expansion of $f(x)$ up to the second order in $(x - x_0)$ gives the contribution to I of the order $O(\hbar^{1/2})$.

2. The formula (15) is exact if the function $f(x)$ is quadratic in x, since in this case the Taylor expansion terminates at the second derivative of $f(x)$.
3. To simplify the notations, let us denote $x - x_0$ by t, and the i'th coefficient in the Taylor series of $f(x)$ — by a_i. The integral (14) becomes

$$I = e^{\frac{i}{\hbar}a_0} \int_{\infty}^{\infty} dt\, e^{\frac{i}{\hbar}\sum_{n=2}^{\infty} a_n t^n} \ . \tag{16}$$

Now we assume that the coefficients $a_n t^n/\hbar$ for $n > 2$ are all much smaller than 1. This allows us to expand the exponents $e^{\frac{i}{\hbar}a_3 t^3}$, $e^{\frac{i}{\hbar}a_4 t^4}$ in the integrand of (16). We have

$$I = e^{\frac{i}{\hbar}a_0} \int_{-\infty}^{\infty} dt\, e^{\frac{i}{\hbar}a_2 t^2} \left(1 + \frac{i}{\hbar}a_3 t^3 + \frac{i}{\hbar}a_4 t^4 - \frac{1}{2\hbar^2}a_3^2 t^6 + ...\right) \ . \tag{17}$$

To compute the next to the leading order (NLO) coefficient in I, it is enough to expand up to t^6 in the r.h.s. of eq. (17). The following formulas can be derived using the definition of the Gamma function or by taking derivatives of a gaussain integral with respect to the constant in the exponent.

$$\int dy\, y^4 e^{\frac{i}{\hbar}ay^2} = \frac{3\sqrt{\pi}}{4\left(-\frac{ia}{\hbar}\right)^{5/2}} \ , \tag{18}$$

$$\int dy\, y^6 e^{\frac{i}{\hbar}ay^2} = \frac{15\sqrt{\pi}}{8\left(-\frac{ia}{\hbar}\right)^{7/2}} \ , \tag{19}$$

we arrive at

$$I = e^{\frac{i}{\hbar}a_0} \sqrt{\frac{\pi i\hbar}{a_2}} \left(1 + \frac{i\left(12a_2a_4 - 15a_3^2\right)\hbar}{16a_2^3} + O(\hbar^2)\right). \tag{20}$$

Hence, the first correction to the leading-order ($O(\hbar^{1/2})$) result (15) is given by the $O(\hbar^{3/2})$-term in eq. (20).
Let us make a comment regarding eq. (20). It suggests that I can be computed as an expansion in powers of $\hbar^{1/2}$. This expansion is valid as soon as $\hbar^{1/2}$ is confined within its radius of convergence. Within this radius, the series whose first two terms we just computed must converge to I. Note, however, that in deriving eq. (20) we assumed $\hbar$ to be *large* enough in order for the expansion of the exponent in eq. (16) to make sense. This rises doubts about legitimacy of the formula (20) in the limit

1. We use the standard formula for the Gaussian integral, assuming that it remains valid when the parameter in the exponent is continued from real to complex values. For the proof of legitimacy of such continuation, see Appendix A.

$\hbar \to 0$. The truth is that the series in the r.h.s. of this formula is, in fact, asymptotic. This means that its radius of convergence is zero and, hence, if we continue to compute higher-order terms in $\hbar^{1/2}$, we will see that from some point such terms start growing, making the full series divergent at any finite $\hbar$. Still, one can extract meaningful information from the asymptotic series. As is proved in asymptotic analysis, the best approximation to the exact answer is provided by taking first several terms of the series. Thus, after all, the result (20) is correct.

4. Here as $f(x)$ we take the function which coincides with $x^2 - x^4$ at $x \geqslant \delta$, and which goes to zero and has no extrema at finite x in the interval $x < \delta$. For example,

$$f(x) = \begin{cases} x^2 - x^4\,, & x \geqslant \delta\,, \\ \dfrac{\delta^4\left(1-\delta^2\right)^3}{(3\delta^3 - 2\delta - 2\delta^2 x + x)^2}\,, & x < \delta\,. \end{cases} \tag{21}$$

Then, the only finite extremum point of $f(x)$ is $x_0 = 1/\sqrt{2}$. We have,

$$\begin{aligned} a_0 &= f(x_0) = \frac{1}{4}\,, & a_2 &= \frac{f''(x_0)}{2} = -2\,, \\ a_3 &= \frac{f'''(x_0)}{6} = -2\sqrt{2}\,, & a_4 &= \frac{f^{(IV)}(x_0)}{24} = -1\,. \end{aligned} \tag{22}$$

Substituting this into eq. (20) gives

$$J = e^{\frac{i}{4\hbar}}\sqrt{\frac{-i\pi\hbar}{2}}\left(1 + \frac{3i\hbar}{4}\right)\,. \tag{23}$$

3. Classical Actions

1. For the free particle, the equation of motion is $\ddot{x} = 0$, and the solution is $x(t) = At + B$, with A, B some constants, which are fixed by the boundary conditions, $x(0) = x_i$, $x(T) = x_f$, so that $x(t) = \frac{x_f - x_i}{T}t + x_i$. Substituting $x(t)$ to the action gives

$$S_{cl} = \int_0^T \frac{m\dot{x}^2}{2}dt = \frac{m}{2}\frac{(x_f - x_i)^2}{T}\,. \tag{24}$$

2. The harmonic oscillator requires a bit more work. The general solution is $x(t) = A\sin\omega t + B\cos\omega t$, where A and B are fixed by the boundary conditions $x(0) = x_i$, $x(T) = x_f$. We have

$$A = \frac{x_f - x_i\cos\omega T}{\sin\omega T}\,, \qquad B = x_i\,. \tag{25}$$

The action reads as follows,

$$S_{cl} = \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt\,, \quad x = x(t)\,. \tag{26}$$

The first term can be integrated by parts,

$$\int_0^T \dot{x}^2 dt = x\dot{x}|_0^T - \int_0^T x\ddot{x}\, dt . \tag{27}$$

For the harmonic oscillator, $\ddot{x} = -\omega^2 x$. Hence

$$\int_0^T \dot{x}^2 dt = x\dot{x}|_0^T + \omega^2 \int_0^T x^2 dt , \tag{28}$$

and

$$S_{cl} = \frac{m}{2}\left(x\dot{x}|_0^T + \omega^2 \int_0^T x^2 dt - \omega^2 \int_0^T x^2 dt \right) = \frac{m}{2}\, x\dot{x}|_0^T \ , \tag{29}$$

or

$$S_{cl} = \frac{m}{2}\left(x_f \dot{x}(T) - x_i \dot{x}(0) \right) . \tag{30}$$

Taking derivatives of $x(t)$, we obtain,

$$\begin{aligned} \dot{x}(0) &= \omega \frac{x_f - x_i \cos\omega T}{\sin\omega T} , \\ \dot{x}(T) &= \omega \frac{x_f - x_i \cos\omega T}{\sin\omega T} \cos\omega T - \omega x_i \sin\omega T . \end{aligned} \tag{31}$$

The result is

$$S_{cl} = \frac{m\omega}{2\sin\omega T}\left((x_i^2 + x_f^2)\cos\omega T - 2x_i x_f \right) . \tag{32}$$

If the trajectory $x(t)$ represents a full period of oscillations, then $x_i = x_f$ and $T = 2\pi/\omega$. Plugging this into eq. (32), we have

$$S_{cl} = m\omega \frac{\cos\omega T - 1}{\sin\omega T}\bigg|_{T=\frac{2\pi}{\omega}} = 0 . \tag{33}$$

3. Let x_1, x_2 be two independent solutions of the homogeneous equation of motion, for example,

$$x_1 = \sin\omega t , \quad x_2 = \cos\omega t . \tag{34}$$

Then, the general solution of the equation with the external force included,

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} , \tag{35}$$

is of the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t) , \tag{36}$$

where $x_p(t)$ is the particular solution of eq. (35). The latter can be obtained by several methods. For example, the method of variation of parameters yields the following particular solution [2]

$$x_p(t) = \frac{1}{m\omega}\left(x_1(t) \int_0^t F(t')x_2(t')dt' - x_2(t) \int_0^t F(t')x_1(t')dt' \right) . \tag{37}$$

2. See, e.g., Carl M. Bender, Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I*, Springer, 1999, p.15.

Solving the boundary values problem, we have in eq. (36),

$$c_1 = \frac{1}{\sin\omega T}(x_f - x_i\cos\omega T - x_p(T)) , \quad c_2 = x_i , \tag{38}$$

where we used the fact that $F(0) = F(T) = 0$.
Next, we compute the action

$$S = \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\, dt + \int_0^T xF(t)\, dt . \tag{39}$$

Substituting eq. (36), we have

$$S_{cl} = S_{cl,F=0} + S_1 + S_2 + S_3 , \tag{40}$$

where

$$\begin{aligned} S_1 &= \int_0^T (c_1 x_1(t) + c_2 x_2(t))F(t)\, dt , \\ S_2 &= \int_0^T x_p(t)F(t)\, dt , \\ S_3 &= \frac{m}{2}\int_0^T (\dot{x}_p(t)^2 - \omega^2 x_p(t)^2)\, dt . \end{aligned} \tag{41}$$

The term $S_{cl,F=0}$ is the action of the undamped oscillator given in eq. (32). The next term is written explicitly with the help of eqs. (34) and (38). S_2 is written by using eqs. (34), (37) and (38). Finally, S_3 is, in fact, equal to $-\frac{1}{2}S_2$. Indeed,

$$\begin{aligned} \frac{m}{2}\int_0^T (\dot{x}_p(t)^2 - \omega^2 x_p(t)^2)\, dt &= -\frac{m}{2}\int_0^T x_p(t)(\ddot{x}_p(t) + \omega^2 x_p(t))\, dt \\ &= -\frac{1}{2}\int_0^T x_p(t)F(t)\, dt = -\frac{1}{2}S_2 . \end{aligned} \tag{42}$$

Overall,

$$\begin{aligned} S_{cl} = \frac{m\omega}{2\sin\omega T}\Bigg[&(x_i^2 + x_f^2)\cos\omega T - 2x_i x_f + \frac{2x_f}{m\omega}\int_0^T F(t)\sin\omega t\, dt \\ &+\frac{2x_i}{m\omega}\int_0^T F(t)\sin\omega(T-t)dt - \frac{2}{m^2\omega^2}\int_0^T\int_0^t F(t)F(t')\sin\omega(T-t)\sin\omega t' dt' dt\Bigg] . \end{aligned} \tag{43}$$

4**. More about completeness of Coherent states

This problem is advanced and outside the main scope of the course. If you have ideas how to solve it, you can share them on the seminar.