QUANTUM PHYSICS III

Solutions to Problem Set 1 23 September 2018

1. Gaussian Integrals

To compute the first integral, one can use the following trick. First, multiply I_1 by itself,

$$I_1^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdy\, e^{-\frac{x^2+y^2}{2}} \;, \tag{1}$$

and then compute the obtained expression using the polar coordinates,

$$I_1^2 = 2\pi \int_0^{\infty} dr\, re^{-\frac{r^2}{2}} = 2\pi \;. \tag{2}$$

Hence,

$$I_1 = \sqrt{2\pi} \;. \tag{3}$$

To compute I_2 and I_3, we first note that

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}} \;, \tag{4}$$

and

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x-x_0)^2} = \int_{-\infty}^{\infty} dx'\, e^{-\frac{1}{2}x'^2} = I_1 \;. \tag{5}$$

So, we have

$$I_2 = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+bx} = \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}\left(x-\frac{b}{a}\right)^2+\frac{b^2}{2a}} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}} \;, \tag{6}$$

$$I_3 = \int_{-\infty}^{\infty} x^2 e^{-\frac{ax^2}{2}} = -2\frac{d}{da}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} = -2\frac{d}{da}\sqrt{\frac{2\pi}{a}} = \frac{1}{a}\sqrt{\frac{2\pi}{a}} \;. \tag{7}$$

2. A Gaussian packet

1. Since the wave function $\Psi(p,0) = \frac{A}{(2\pi)^{1/4}} e^{-\frac{\sigma^2}{\hbar^2}(p-p_0)^2}$ must be normalized to one, it follows that

$$\int_{-\infty}^{\infty} |\Psi(p,0)|^2 dp = 1 \;\Rightarrow\; A = \sqrt{\frac{2\sigma}{\hbar}} \;. \tag{8}$$

2. Recall that by definition the Fourier image is (note our convention about 2π multipliers)

$$\Psi(x,0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dp\, \Psi(p,0) e^{\frac{i}{\hbar} px} \,. \tag{9}$$

Substituting the expression for $\Psi(p,0)$, we obtain,

$$\Psi(x,0) = \frac{\sqrt{2\sigma/\hbar}}{(2\pi)^{3/4}} \int_{-\infty}^{\infty} dp\, e^{-\frac{\sigma^2 p^2}{\hbar^2} + \left(\frac{2\sigma^2 p_0}{\hbar} + ix\right)\frac{p}{\hbar}} e^{-\frac{\sigma^2 p_0^2}{\hbar^2}} = \left(\frac{\hbar^2}{2\pi\sigma^2}\right)^{1/4} e^{\frac{i}{\hbar} p_0 x} e^{-\frac{x^2}{4\sigma^2}} \,. \tag{10}$$

Thus, $\Psi(x,0)$ is again a Gaussian function.
The dispersion of the operator A is defined as

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \,. \tag{11}$$

Let us first compute $\Delta p(0)$. We have

$$\langle p(0) \rangle = \int_{-\infty}^{\infty} dp\, p |\Psi(p,0)|^2 = \int_{-\infty}^{\infty} dp\, (p + p_0) |\Psi(p + p_0, 0)|^2 = p_0 \,, \tag{12}$$

because the expression $p\, |\Psi(p + p_0, 0)|^2$ is an odd function of p. Hence,

$$(\Delta p(0))^2 = \int_{-\infty}^{\infty} dp\, p^2 |\Psi(p,0)|^2 - p_0^2 = \int_{-\infty}^{\infty} dp\, p^2 |\Psi(p + p_0, 0)|^2 + p_0^2 - p_0^2 = \hbar^2/4\sigma^2 \,, \tag{13}$$

and

$$\Delta p(0) = \hbar/2\sigma \,. \tag{14}$$

Similarly, using the expression (10) for $\Psi(x,0)$, one obtains

$$\Delta x(0) = \sigma \,. \tag{15}$$

Of course, the above expressions for the dispersions can be seen immediately from the expressions for $\Psi(p,0)$ and $\Psi(x,0)$, since the latter are Gaussian functions. Note that $\Delta x(0) \Delta p(0) = \hbar/2$, that is, the state $|\Psi\rangle$ minimizes the uncertainty relation for the pair of operators $x(0)$, $p(0)$.

3. Solving the Cauchy problem gives

$$\Psi(p,t) = \Psi(p,0) e^{-\frac{i}{\hbar}\omega(p)t} \,, \quad \omega(p) = \frac{p^2}{2m} \,. \tag{16}$$

To find $\Psi(x,t)$, we make the Fourier transform of $\Psi(p,t)$ and use the results of exercise 1. We have

$$\begin{aligned}
\Psi(x,t) &= \frac{\sqrt{2\sigma}}{(2\pi)^{3/4}} \int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2}\left(2\sigma^2 + \frac{i\hbar t}{2m}\right)\frac{p^2}{\hbar^2} + \left(\frac{\sigma^2 p_0}{\hbar} + ix\right)\frac{p}{\hbar}} e^{-\frac{\sigma^2 p_0^2}{\hbar^2}} \\
&= \frac{\sqrt{2\sigma}}{(2\pi)^{3/4}} \sqrt{\frac{2\pi}{2\sigma^2 + \frac{i\hbar t}{m}}} \exp\left[\frac{\left(ix + \frac{\sigma^2}{\hbar} p_0\right)^2}{4\sigma^2 + \frac{2i\hbar t}{m}}\right] e^{-\frac{\sigma^2}{\hbar^2} p_0^2} \\
&= \frac{\sqrt{2\sigma}}{(2\pi)^{1/4}} \sqrt{\frac{2\sigma^2 - \frac{i\hbar t}{m}}{\sigma^2 + \frac{\hbar^2 t^2}{m^2}}} \exp\left[-\frac{\left(x - \frac{p_0 t}{m}\right)^2}{4\sigma^2 + \frac{2i\hbar t}{m}}\right] e^{\frac{i}{\hbar} x p_0} e^{-\frac{i t p_0^2}{2m\hbar}} \\
&= \left(\frac{8\sigma^2}{\pi}\right)^{1/4} \frac{\exp\left[-i\left(\frac{1}{2}\arctan\frac{\hbar t}{2m\sigma^2} + \frac{p_0^2}{2m\hbar} t\right)\right]}{\left(16\sigma^4 + \frac{4\hbar^2 t^2}{m^2}\right)^{1/4}} e^{\frac{i}{\hbar} p_0 x} e^{-\frac{\left(x - \frac{p_0}{m} t\right)^2}{4\sigma^2 + \frac{2i\hbar t}{m}}} \,,
\end{aligned} \tag{17}$$

where at the last step we used the fact that $\sqrt{c} = |c|^{1/2} e^{\frac{i}{2}\arg c}$ for a complex c.
From eqs. (16) and (17) it follows that

$$\Delta p(t) = \Delta p(0) \ , \quad \Delta x(t) = \Delta x(0) \sqrt{1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}} \ . \tag{18}$$

When t increases, $\Delta x(t)$ grows as well, and the equality $\Delta x \Delta p = \hbar/2$ is not valid anymore : the wave packet is spreading. From eq. (17) we note also that p_0/m is nothing but the group velocity of the wave packet.

3. Quantum fluctuations

We assume that the system comprising the hill and the object on the top of it does not interact with anything. This allows us to drop out the potential term in the Schrödinger equation. The wave function of the object can now be taken Gaussian, like in the previous exercise. On physical grounds we expect that the probability density for the object to fall from the top of the hill is saturated at the time when its dispersion $\Delta x(t)$ becomes equal to the size of the top l. The dispersion is given by (see eq. (18))

$$(\Delta x(t))^2 = \sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2} \ . \tag{19}$$

1. Denote the fall time by t_0. Consider the equality $\Delta x(t_0) = l$ as an equation for $t_0 = t_0(\sigma)$:

$$\left(\frac{\hbar t_0}{2m}\right)^2 = -\sigma^4 + l^2\sigma^2. \tag{20}$$

   Maximizing t_0 with respect to σ gives

$$\sigma = \frac{l}{\sqrt{2}} \ , \quad t_{0\,max} = \frac{ml^2}{\hbar} \sim \frac{1g \cdot 1cm^2}{10^{-33}\frac{g \cdot cm^2}{s}} = 10^{33} s \ . \tag{21}$$

2. Substituting $\sigma = 10^{-9}cm \ll l$ into eq. (20) gives

$$t_0 \approx \frac{ml\sigma}{\hbar} = 10^{18} s \ . \tag{22}$$

4. Harmonic oscillator

1. Recall that $H = \hbar\omega(a^\dagger a + 1/2)$ and $[a, a^\dagger] = 1$. We have

$$\begin{aligned} [a, H] = \hbar\omega[a, a^\dagger a + 1/2] &= \hbar\omega[a, a^\dagger a] = \hbar\omega\left(aa^\dagger a - a^\dagger aa\right) \\ &= \hbar\omega[a, a^\dagger]a = \hbar\omega a \ . \end{aligned} \tag{23}$$

   Similarly, $[a^\dagger, H] = -\hbar\omega a^\dagger$.
   In the Heisenberg picture,

$$i\hbar\frac{da}{dt} = [a, H] = \hbar\omega a \quad \Rightarrow \quad a(t) = a(0)e^{-i\omega t} \ . \tag{24}$$

Therefore,

$$[a(t), a^\dagger(t)] = [a(0)e^{-i\omega t}, a^\dagger(0)e^{i\omega t}] = [a(0), a^\dagger(0)] = 1 \ . \qquad (25)$$

2. One should express $x(t)$ and $p(t)$ through $\alpha(t)$ and $\alpha^*(t)$,

$$\begin{aligned} x(t) &= \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha(t) + \alpha^*(t)) \ , \\ p(t) &= -i\sqrt{\frac{\hbar m\omega}{2}}\,(\alpha(t) - \alpha^*(t)) \ , \end{aligned} \qquad (26)$$

and put them into H,

$$\begin{aligned} H = \frac{1}{2m}p(x)^2 + \frac{1}{2}m\omega^2 x(t)^2 &= \frac{1}{2m}p(0)^2 + \frac{1}{2}m\omega^2 x(0)^2 \\ &= \hbar\omega\frac{1}{4}4\alpha(0)\alpha^*(0) = \hbar\omega|\alpha(0)|^2 \ , \end{aligned} \qquad (27)$$

where we made use of the conservation of H with time.

5. Gaussian integrals in more dimensions

1. The idea is to diagonalize the matrix A in order to reduce the integral into the product of integrals of Gaussian functions. Let O be the desired orthogonal transformation,

$$O^{-1}AO = \mathrm{diag}(\lambda_1, ..., \lambda_N) \ , \qquad (28)$$

where λ_i are eigenvalues of A. The corresponding change of variables reads as follows,

$$y = O^t x \quad \Rightarrow \quad dy_1...dy_N = \det O \cdot dx_1...dx_N = dx_1...dx_N \ , \qquad (29)$$

since O is orthogonal. Applying the transformation (29) to the integral gives

$$\begin{aligned} &\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1...dx_N \ \exp\left(-\frac{1}{2}x^t A x + B^t x\right) \\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dy_1...dy_N \ \exp\left(-\frac{1}{2}y^t\left(O^{-1}AO\right)y + B^t O y\right) \\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dy_1...dy_N \ \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\lambda_i y_i^2 + B^t O y\right) \\ &= \sqrt{\frac{(2\pi)^N}{\Pi_{i=1}^N \lambda_i}}\exp\left(\frac{1}{2}\sum_{i=1}^{N} B_j O_{ji}\lambda_i^{-1}(O^{-1})_{ik}B_k\right) = \sqrt{\frac{(2\pi)^N}{\det A}}\exp\left(\frac{1}{2}B^t A^{-1} B\right) \ . \end{aligned} \qquad (30)$$

2. Here one can use the following trick. First, given the exponent $\exp\left(-\frac{1}{2}x^t A x\right)$, we supplement it with the "source" $\exp(B^t x)$ of the variable x. Then we differentiate the source with respect to B_i to obtain the factor x_i in the integrand. Finally, at

the end of calculation we take the limit $B = 0$. Here is the implementation of this program :

$$
\begin{aligned}
&\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1 ... dx_N \; x_{i_1} x_{i_2} \exp\left(-\frac{1}{2} x^t A x\right) \\
&= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1 ... dx_N \; x_{i_1} x_{i_2} \; \exp\left(-\frac{1}{2} x^t A x + B^t x\right)\bigg|_{B=0} \\
&= \frac{d}{dB_{i_1}} \frac{d}{dB_{i_2}} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1 ... dx_N \;\; \exp\left(-\frac{1}{2} x^t A x + B^t x\right)\bigg|_{B=0} \\
&= \sqrt{\frac{(2\pi)^N}{\det A}} \frac{d}{dB_{i_1}} \frac{d}{dB_{i_2}} \; \exp\left(\frac{1}{2} B_i \left(A^{-1}\right)_{ij} B_j\right)\bigg|_{B=0} \; = \; \sqrt{\frac{(2\pi)^N}{\det A}} \left(A^{-1}\right)_{i_1 i_2} \; .
\end{aligned}
\tag{31}
$$

Thus,

$$
\langle x_{i_1} x_{i_2} \rangle = \left(A^{-1}\right)_{i_1 i_2} \; . \tag{32}
$$

As for the average $\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle$, it is obtained by simply adding more differentials d/dB_{i_p} with various i_p, and the result

$$
\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle = \langle x_{i_1} x_{i_2} \rangle \langle x_{i_3} x_{i_4} \rangle + \langle x_{i_1} x_{i_3} \rangle \langle x_{i_2} x_{i_4} \rangle + \langle x_{i_1} x_{i_4} \rangle \langle x_{i_2} x_{i_3} \rangle \tag{33}
$$

is reproduced straightforwardly. Note finally that any odd number of derivatives inevitably gives some B_{i_p} appearing before the exponent, hence, after setting $B = 0$, all such terms vanish, and

$$
\langle x_{i_1} x_{i_2} ... x_{i_k} \rangle = 0 \; , \quad \text{if } k \text{ is odd} \; . \tag{34}
$$