EPFL Automne 2018
Geometrie, MATH-120

Solutions série 3

Exercice 3. (Theoreme de l'hypothenuse) Soient $P \neq Q$ deux points et $\mathcal{C}$ let cercle de centre le milieu de $[PQ]$ et de rayon $d(P,Q)/2$. Montrer que pour tout point $R \in \mathcal{C}$ le triangle $[PQR]$ est rectangle en R.

Solution 3. We will denote by M the midpoint of the segment $[PQ]$. We want to show that $\langle \vec{RP}, \vec{RQ} \rangle = 0$. We have $\vec{RP} = \vec{MR} + \vec{PM}$ and $\vec{RQ} = \vec{RM} + \vec{MQ}$ and therefore

$$\begin{aligned}\langle \vec{RP}, \vec{RQ} \rangle &= \langle \vec{MR} + \vec{PM}, \vec{RM} + \vec{MQ} \rangle \\ &= \langle \vec{MR}, \vec{RM} \rangle + \langle \vec{MR}, \vec{MQ} \rangle + \langle \vec{PM}, \vec{RM} \rangle + \langle \vec{PM}, \vec{MQ} \rangle \\ &= -||\vec{MR}||^2 + \langle \vec{MR}, \vec{MQ} \rangle + \langle \vec{PM}, \vec{RM} \rangle + ||\vec{PM}||^2,\end{aligned}$$

using that $\vec{PM} = \vec{MQ}$. Since the vector $\vec{MR}$ has the same norm as $\vec{PM}$ we obtain

$$\begin{aligned}\langle \vec{RP}, \vec{RQ} \rangle &= \langle \vec{MR}, \vec{MQ} \rangle - \langle \vec{PM}, \vec{MR} \rangle \\ &= \langle \vec{MR}, \vec{MQ} - \vec{PM} \rangle = 0 \\ &= \langle \vec{MR}, \vec{0} \rangle = 0,\end{aligned}$$

as desired.

Exercice 5. Soient $\vec{u}, \vec{v} \in \mathbb{R}^2$ deux vecteurs non-nuls et orthogonaux ($\langle \vec{u}, \vec{v} \rangle = 0$).

On a vu que tout vecteur $\vec{w}$ s'ecrit de maniere unique sous la forme

$$\vec{w} = \alpha \vec{u} + \beta \vec{v}, \ \alpha, \beta \in \mathbb{R}$$

avec des expression explicites pour α et β.

1. Soit $\mathrm{Proj}_{\vec{v}} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ l'application

   $$\mathrm{Proj}_{\vec{v}} : \vec{w} \mapsto \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}.$$

   Montrer que $\mathrm{Proj}_{\vec{v}}$ est lineaire : $\forall \lambda \in \mathbb{R}, \ \vec{w}, \vec{w}' \in \mathbb{R}^2$

   $$\mathrm{Proj}_{\vec{v}}(\lambda.\vec{w} + \vec{w}') = \lambda. \mathrm{Proj}_{\vec{v}}(\vec{w}) + \mathrm{Proj}_{\vec{v}}(\vec{w}').$$

   calculer son image et son noyau. Calculer $\mathrm{Proj}_{\vec{v}} \circ \mathrm{Proj}_{\vec{v}}$.

2. Montrer que $\mathrm{Proj}_{\vec{v}}$ ne depend en fait que de la droite $(\vec{v}) = \mathbb{R}.\vec{v}$ et non du vecteur (non-nul) $\vec{v}$ contenu dans cette droite. On appelle-t-on $\mathrm{Proj}_{\vec{v}}$ la *projection orthogonale sur la droite* $(\vec{v})$, pourquoi ?
3. Soit $\mathrm{sym}_{\vec{u}} : \mathbb{R}^2 \mapsto \mathbb{R}^2$, l'application

$$\mathrm{sym}_{\vec{u}} : \vec{w} \mapsto \vec{w} - 2.\,\mathrm{Proj}_{\vec{v}}(\vec{w}).$$

4. Montrer que $\mathrm{sym}_{\vec{u}}$ est une isometrie lineaire et calculer

$$\mathrm{sym}_{\vec{v}} \circ \mathrm{sym}_{\vec{v}} = \mathrm{Id}_{\mathbb{R}^2}.$$

5. Montrer que $\mathrm{sym}_{\vec{u}}$ ne depend en fait que de la droite $(\vec{u}) = \mathbb{R}.\vec{u}$ et non du vecteur (non-nul) $\vec{u}$ contenu dans cette droite. On appelle-t-on $\mathrm{sym}_{\vec{v}}$ la *symetrie orthogonale par rapport a l'axe* $(\vec{u})$, pourquoi ?

Solution 5. For the ease of typing we will drop the arrow from the notation $\vec{v}$ and just write v.

1. We omit the proof that Proj_v is linear. To show that $\mathrm{Proj}_v \circ \mathrm{Proj}_v = \mathrm{Proj}_v$ we take an arbitrary $x \in \mathbb{R}^2$ and calculate

$$\begin{aligned} ||v||^2\,\mathrm{Proj}_v(\mathrm{Proj}_v(x)) &= \langle \mathrm{Proj}_v(x), v\rangle v \\ &= \left\langle \frac{\langle x, v\rangle}{||v||^2} v, v \right\rangle v = \langle x, v\rangle v. \end{aligned}$$

Now we divide by $||v||^2$ to obtain the result. As for the kernel, note that for any $x \in \mathbb{R}^2$ we have

$$\mathrm{Proj}_v(x) = 0 \quad \Leftrightarrow \quad \frac{\langle x, v\rangle}{||v||^2} v = 0 \quad \Leftrightarrow \quad \langle x, v\rangle = 0,$$

since $v \neq 0$. This shows that the kernel of Proj_v is the subspace $\{x \in \mathbb{R}^2 : \langle x, v\rangle = 0\}$ which is sometimes denoted $(\mathbb{R}v)^\perp$ and called the orthogonal complement of v. It is clear that the image of Proj_v is contained in the line $\mathbb{R}v$. The identity $\mathrm{Proj}_v(tv) = tv$, which holds for all $t \in \mathbb{R}$, shows that reverse inclusion also holds, hence $\mathrm{Proj}_v(\mathbb{R}^2) = \mathbb{R}v$.

2. What we need to show is this : For all $t \in \mathbb{R}$ we have $\mathrm{Proj}_{tv} = \mathrm{Proj}_v$. To prove this, we take any $x \in \mathbb{R}^2$ and compute

$$\mathrm{Proj}_{tv}(x) = \frac{1}{||tv||^2}\langle x, tv\rangle tv = \frac{t^2}{t^2||v||^2}\langle x, v\rangle v = \mathrm{Proj}_v(x).$$

3. Nothing was asked here.

4. The map sym_v is a linear combination of linear maps (by part 1), hence linear. Therefore, to show that sym_u is an isometry, it suffices to verify that $||\mathrm{sym}_v(x)|| = ||x||$ for all $x \in \mathbb{R}^2$ [1] . We use the identity $||a-b||^2 = ||a||^2 - 2\langle a, b\rangle + ||b||^2$ with $a = x$ and $b = 2\,\mathrm{Proj}_v(x)$ and expand

$$\begin{aligned} ||\mathrm{sym}_u(x)||^2 &= ||x||^2 - 4\langle x, \frac{\langle x, v\rangle}{||v||^2} v\rangle + 4||\frac{\langle x, v\rangle}{||v||^2} v||^2 \\ &= ||x||^2 - 4\frac{\langle x, v\rangle^2}{||v||^2} + 4\frac{\langle x, v\rangle^2 ||v||^2}{||v||^4} = ||x||^2, \end{aligned}$$

giving the desired equality by taking square roots. We now show that $\mathrm{sym}_v \circ \mathrm{sym}_v = \mathrm{Id}_{\mathbb{R}^2}$. Abbreviate $p = \mathrm{Proj}_v$, $s = \mathrm{sym}_v$ and $1 = \mathrm{Id}_{\mathbb{R}^2}$. Addition and composition of linear maps in $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ behave just like the usual addition and multiplication of numbers – except that composition of maps is not commutative. This justifies the following equalities of linear maps

$$s \circ s = (1-2p) \circ (1-2p) = 1 - 2p - 2p + 4p^2 = 1 - 4p + 4p = 1,$$

where we used that $p^2 = p$ (and nothing more about p).

5. From part 2 it is clear that sym_v only depends on the line $\mathbb{R}v$ because the projection Proj_v only depends on the line $\mathbb{R}v$ (and the identity map doesn't depend on any choices, of course). Geometrically, sym_v is the reflection along the line orthogonal to v.

Exercice 6. Soit $a, b, c \in \mathbb{R}$ tels que $\Delta = b^2 - 4ac < 0$ et $a > 0$. On definit le polynome (homogene) de degre 2

$$Q(X, Y) = aX^2 + bXY + cY^2.$$

On obtient donc une fonction

$$Q : \begin{array}{ccc} \mathbb{R}^2 & \mapsto & \mathbb{R} \\ \vec{u} = (x, y) & \mapsto & Q(P) = Q(x, y) \end{array}$$

(ici un vecteur $\vec{u}$ du plan est repere par ces coordonnees dans la base canonique.) On defini une application

$$\langle \cdot, \cdot \rangle_Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}.$$

par

$$\langle \vec{u}, \vec{v}\rangle_Q := \frac{1}{4}(Q(\vec{u} + \vec{v}) - Q(\vec{u} - \vec{v})).$$

1. Que vaut $\langle \vec{u}, \vec{u}\rangle_Q$?

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1. because then we have $d(\mathrm{sym}_v(x), \mathrm{sym}_v(y)) = ||\mathrm{sym}_v(x) - \mathrm{sym}_v(y)|| = ||\mathrm{sym}_v(x-y)|| = ||x-y|| = d(x, y)$ for all $x, y \in \mathbb{R}^2$.

2. Montrer que $\langle\cdot,\cdot\rangle_Q$ est un produit scalaire (defini-positif) sur $\mathbb{R}^2$. On pose alors

$$\|\cdot\|_Q : \vec{u} \in \mathbb{R}^2 \mapsto \langle\vec{u},\vec{u}\rangle_Q^{1/2}$$

et

$$d_Q(\cdot,\cdot) : (R,S) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \|\vec{RS}\|_Q.$$

3. Montrer que inegalite de Cauchy-Schwarz est vraie :

$$\forall \vec{u},\vec{v} \in \mathbb{R}^2,\ |\langle\vec{u},\vec{v}\rangle_Q| \leqslant \|\vec{u}\|_Q\|\vec{v}\|_Q$$

avec egalite si et seulement si $\vec{u}$ et $\vec{v}$ sont proportionnels.

4. Montrer que d_Q definit une distance sur $\mathbb{R}^2$. Si $a=b=c=1$ dessiner la boule unite $B(0,1)_Q$.
5. Dans le cas general, etant donne deux points R,S, on definit le "segment" relativement a la distance d_Q par

$$[R,S]_Q = \{T \in \mathbb{R}^2 |\ d_Q(R,T) + d_Q(T,S) = d_Q(R,S)\}.$$

Quelle est la forme de ce Q-segment ?

6. Montrer que pour tout $\vec{u} \neq \vec{0}$, il existe $\vec{v} \neq 0$ tel que $\langle\vec{u},\vec{v}\rangle_Q = 0$ (on dira que $\vec{u},\vec{v}$ sont orthogonaux pour le produit scalaire $\langle\cdot,\cdot\rangle_Q$. Montrer qu'alors pour tout $\vec{w} \in \mathbb{R}^2$ il existe $\lambda,\mu \in \mathbb{R}$ tels que

$$\vec{w} = \lambda\vec{u} + \mu\vec{v}$$

et calculer λ,μ en fonction du produit scalaire $\langle\cdot,\cdot\rangle_Q$.

Solution 6. We omit the decorating arrow in the notation for a vector.

1. It follows immediately from the definitions that $Q(tv) = t^2Q(v)$ for all $t \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ and it is also clear that $Q(0) = 0$. Therefore we have

$$\begin{aligned}\langle u,u\rangle_Q &= \tfrac{1}{4}\left(Q(u+u) - Q(u-u)\right) = \tfrac{1}{4}\left(Q(2u) - Q(0)\right)\\ &= \tfrac{1}{4}(4Q(u)) = Q(u).\end{aligned}$$

2. To prove that $\langle\cdot,\cdot\rangle_Q$ is symmetric and bilinear, we express for general $u_1 = \begin{pmatrix}x_1\\y_1\end{pmatrix}$, $u_2 = \begin{pmatrix}x_2\\y_2\end{pmatrix} \in \mathbb{R}^2$ the inner product

$$\begin{aligned}\langle u_1,u_2\rangle_Q &= \frac{1}{4}\left(Q(\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}) - Q(\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix})\right)\\ &= \frac{1}{4}(a(x_1+x_2)^2 + (x_1+x_2)(y_1+y_2) + c(y_1+y_2)^2\\ &\qquad - a(x_1-x_2)^2 - b(x_1-x_2)(y_1-y_2) - c(y_1-y_2)^2)\\ &= ax_1x_2 + \frac{b}{2}(x_1y_2+x_2y_1) + cy_1y_2.\end{aligned}$$

This makes it clear that $\langle u_1, u_2\rangle_Q = \langle u_2, u_1\rangle_Q$ and the bilinearity now follows from

$$\begin{aligned}\langle u_1 + tu_1', u_2\rangle_Q &= a(x_1 + tx_1')x_2 + \frac{b}{2}((x_1 + tx_1')y_2 + x_2(y_1 + ty_1')) + c(y_1 + ty_1')y_2 \\ &= ax_1x_2 + \frac{b}{2}(x_1y_2 + x_2y_1) + cy_1y_2 \\ &\quad t(ax_1'x_2 + \frac{b}{2}(x_1'y_2 + x_2y_1') + cy_1'y_2) \\ &= \langle u_1, u_2\rangle_Q + t\langle u_1', u_2\rangle_Q.\end{aligned}$$

To show that $\langle , \rangle_Q$ is positive definite we complete the square in the definition of Q and write

$$\begin{aligned}Q(x, y) &= ax^2 + bxy + cy^2 \\ &= a(x^2 + \tfrac{b}{a}yx) + cy^2 \\ &= a(x + \tfrac{b}{2a}y)^2 - a(\tfrac{b}{2a}y)^2 + cy^2 \\ &= a(x + \tfrac{b}{2a}y)^2 + (c - \tfrac{b^2}{4a})y^2 \\ &= a(x + \tfrac{b}{2a}y)^2 + (\tfrac{4ac-b^2}{4a})y^2.\end{aligned}$$

Since we are assuming that $a > 0$ and $4ac - b^2 > 0$, we always have $Q(x, y) \geqslant 0$. Suppose that $Q(x, y) = 0$, then we must have

$$a(x + \tfrac{b}{2a}y)^2 = 0 \quad \text{and} \quad (\tfrac{4ac-b^2}{4a})y^2 = 0$$

This implies that $y = 0$ and then $x = 0$ (again because $a > 0$ and $4ac - b^2 > 0$).

3. The proof of the Cauchy-Schwarz inequality given in the lecture works for all inner products : One considers the non-negative polynomial function $P(t) = ||tu + v||_Q^2$ of one real variable $t \in \mathbb{R}$ and and deduces from fact that its discriminant has to be $\leqslant 0$ the Cauchy-Schwarz inequality. (The discriminant can't be > 0 because otherwise P would have two distinct real roots and thus would assume strictly negative values as well).

   We give a second proof using orthogonal projections. These can be defined for all inner products on $\mathbb{R}^2$, in particular for $\langle \cdot, \cdot\rangle_Q$. When proving $|\langle u, v\rangle_Q| \leqslant ||u||_Q||v||_Q$ we can assume that $v \neq 0$ (because for $v = 0$, the inequality becomes $0 \leqslant 0$). Think of v as being fixed. We write an arbitrary u as a sum of its orthogonal projection onto $\mathbb{R}v$ and an other vector :

$$u = \operatorname{Proj}_v(u) + (u - \operatorname{Proj}_v(u)).$$

   A small computation shows that $\langle v, u - \operatorname{Proj}_v(u)\rangle = 0$, implying $\langle \operatorname{Proj}_v(u), u - \operatorname{Proj}_v(u)\rangle = 0$. We may think of $u - \operatorname{Proj}_v(u)$ as the orthogonal projection of u onto a line orthogonal to the line $\mathbb{R}v$. Using this orthogonality we get

$$||u||_Q^2 = ||\operatorname{Proj}_v(u)||_Q^2 + ||u - \operatorname{Proj}_v(u)||_Q^2.$$

Now we just use that $||u - \mathrm{Proj}_v(u)||_Q^2 \geqslant 0$ and obtain

$$||u||_Q^2 \geqslant ||\,\mathrm{Proj}_v(u)||_Q^2 = \frac{\langle u, v\rangle^2}{||v||_Q^4}||v||_Q^2,$$

which rearranges to give the Cauchy-Schwarz inequality. Furthermore, if the vectors are not proportional, then u does not belong to the line $\mathbb{R}v$, which we proved is the image of Proj_v. In particular, $u \neq \mathrm{Proj}_v(u)$ and so $||u - \mathrm{Proj}_v(u)||_Q^2 > 0$ and we see that the inequality becomes strict.

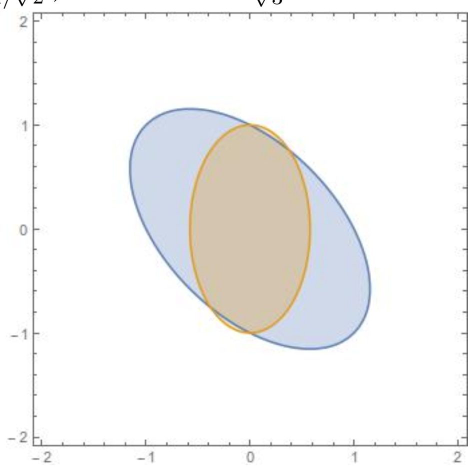
4. In general, given $A, B > 0$ we know how the set $\{(x, y) \in \mathbb{R}^2 : Ax^2 + By^2 \leqslant 1\}$ looks like : it is a filled ellipse with center at the origin, intersecting the coordinate axes at the points $(\pm A^{-1/2}, 0)$ and $(0, \pm B^{-1/2})$. We can reduce the general case to this case by finding a linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that the quadratic from $(Q \circ \varphi)(x, y)$ has no mixed terms xy. We can take the map $\varphi(x, y) := (x - y, x + y)$, because

$$Q(x - y, x + y) = (x - y)^2 + (x - y)(x + y) + (x + y)^2 = 3x^2 + y^2.$$

In principle one can find a suitable φ by writing down a general linear map $\varphi(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$ with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ to obtain a condition on $\alpha, \beta, \gamma, \delta$. (A more enlightening method will be taught in linear algebra at some point). Define

$$\begin{aligned} E &:= \{(x, y) \in \mathbb{R}^2 : 3x^2 + y^2 \leqslant 1\} \\ B &:= \{(x, y) \in \mathbb{R}^2 : Q(x, y) = x^2 + xy + y^2 \leqslant 1\} \end{aligned}$$

By definition, we have $\varphi(E) = B$. So we only need to figure out what the linear map φ does geometrically. We can write $\varphi(v) = \sqrt{2}\psi(v)$, where $\psi(v) = \frac{1}{\sqrt{2}}\varphi(v)$. Then ψ takes the standard orthonormal basis of $\mathbb{R}^2$ to the orthonormal basis $\binom{1/\sqrt{2}}{1/\sqrt{2}}, \binom{-1/\sqrt{2}}{1/\sqrt{2}}$ of $\mathbb{R}^2$ and we see that ψ is a counter-clockwise rotation with angle $\pi/2$. The map φ is therefore a rotation, followed by a scaling with factor $\sqrt{2}$ and we see that B is a filled ellipse centered at the origin, intersecting its axes $\mathbb{R}\binom{1/\sqrt{2}}{1/\sqrt{2}}$ and $\mathbb{R}\binom{-1/\sqrt{2}}{1/\sqrt{2}}$ at distances $\frac{\sqrt{2}}{\sqrt{3}}$ and $\sqrt{2}$ respectively.



5. Define $\mathcal{S} := \{R + t(S - R) \, : \, t \in [0, 1]\}$. We claim that

$$[R, S]_Q = \mathcal{S}.$$

The containment $\mathcal{S} \subseteq [R, S]_Q$ holds because for any $t \in [0, 1]$ we have

$$\begin{aligned} d_Q(R, R + t(S - R)) + d_Q(R + t(S - R), S) = \\ &= ||R + t(S - R) - R||_Q + ||S - (R + t(S - R))||_Q \\ &= t||S - R||_Q + (1 - t)||S - R||_Q \\ &= ||S - R||_Q = d_Q(R, S). \end{aligned}$$

Notice that we have used here that $t \in [0, 1]$ when pulling the scalar $1 - t$ out of the norm. Now let $P \in [R, S]_Q$. We want to show that $P \in \mathcal{S}$.

We first try to show that P belongs to line passing through R and S. To do that, we have to show that $P - R$ is proportional to $S - R$, i.e. that

$$|\langle P - R, S - R\rangle_Q| = ||P - R||_Q||S - R||_Q,$$

using the Cauchy Schwarz inequality theorem. We have information about distances and not inner products. So to bring distances back into the argument we could try to use the formula $\langle a, b\rangle_Q = \frac{1}{2}\left(Q(a + b) - Q(a) - Q(b)\right)$. Doing so with $a = R - P$ and $b = S - R$ gives (dropping henceforth the subscript Q from the notations)

$$\begin{aligned} \langle R - P, S - R\rangle &= \frac{1}{2}\left(||(R - P) + (S - R)||^2 - ||R - P||^2 - ||S - R||^2\right) \\ &= \frac{1}{2}\left(||S - P||^2 - ||R - P||^2 - ||S - R||^2\right) \\ &= \frac{1}{2}\left((||S - R|| - ||R - P||)^2 - ||R - P||^2 - ||S - R||^2\right) \\ &= -||S - R|| \, ||R - P||, \end{aligned}$$

using the assumption $P \in \mathcal{S}$ in the third step. By taking absolute values we get the desired equality.

So now we know that P must at least belong to the the line passing through R and S, which means that we can write $P = R + t(S - R)$ for some real number $t \in \mathbb{R}$. The condition $d_Q(R, P) + d_Q(P, S) = d_Q(R, S)$ tells us that

$$(|t| + |1 - t|)||R - S|| = ||R - S||$$

and hence $|t| + |1 - t| = 1$. This forces $t \in [0, 1]$ and finishes the argument.

Interestingly enough we have in the end reduced the original problem to the problem of showing $|t| + |1 - t| = 1 \Rightarrow t \in [0, 1]$, which is the one-dimensional analogue of the exercise. Notice that we have also provided the solution for the first part of Exercise 2, because the proof works for all inner products.

6. Let u be a nonzero vector. Take any vector $x \notin \mathbb{R}u$. We have seen in our second proof of the Cauchy-Schwarz inequality (in part 3) that the vector $v := x - \mathrm{Proj}_u(x)$ is not zero and orthogonal to u. Since u and v are orthogonal and both nonzero they are linearly independent and therefore they must span $\mathbb{R}^2$. We can give a formula for the numbers λ and μ since if $w = \lambda u + \mu v$ then

$$\langle u, w \rangle = \langle u, \lambda u + \mu v \rangle = \lambda ||u||^2 + \mu \langle u, v \rangle = \lambda ||u||^2$$

and so $\lambda = \frac{\langle u,w \rangle}{||u||^2}$ and similarly $\mu = \frac{\langle v,w \rangle}{||v||^2}$. Thus, we see that the decomposition of w is given by $w = \mathrm{Proj}_u(w) + \mathrm{Proj}_v(w)$.