Exercise 8.

a. $\chi^{(3)}_{ijkl} \sim (\underbrace{\delta_{ij}\delta_{kl}}_{\substack{\text{first 2} \\ + \\ \text{last 2}}} + \underbrace{\delta_{ik}\cdot\delta_{jl}}_{\substack{1,3 \text{ and} \\ 2,4}} + \underbrace{\delta_{il}\cdot\delta_{jk}}_{\substack{1,4 \text{ and} \\ 2,3}})$ are identical

for χ_{1111} all three δ pairs are true and $1 \dashrightarrow 3$; same for $\chi_{1111} : \chi_{2222}; \chi_{3333}$
for any of the other combinations just one $\delta\cdot\delta$ set is non-zero, but
for any combination that has ~~jto~~ an odd number of elements $\chi_{ijkl} \rightarrow 0$.

b. For an isotropic medium that looks the same in every direction the $\chi^{(3)}$ tensor should also be isotropic $\dashrightarrow$ we need an even number of X, Y and z indices

E.g. χ_{zxxx} : 3 X polarized fields emit in the z direction;
but since the material is isotropic there should be an equal $-z$ response.

c. χ_{xxzz} : remains unchanged if we apply the inversion symmetry operator, or a mirror operation, or a rotation operation.
These off-diagonal elements are symmetric - non-vanishing under all symmetry operations of an a isotropic material.
Compare this to a product of two $\chi^{(1)}$ tensors.