QUANTUM PHYSICS III

Solutions to Problem Set 5

19 October 2018

1. On validity of the leading-order (LO) WKB approximation

1. Substituting the wave function

$$\psi = e^{\frac{i}{\hbar}S} \ , \quad S = S_0 + \frac{\hbar}{i}S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + ... \tag{1}$$

into the Schroedinger equation, we get

$$-i\hbar S'' + S'^2 = p^2 \ . \tag{2}$$

Expanding this up to $\mathcal{O}(\hbar^2)$ and equating the $\hbar^2$-terms, we obtain

$$S_1'' + 2S_0'S_2' + S_1^2 = 0 \ \Rightarrow \ S_2' = -\frac{1}{2S_0'}(S_1'' + S_1'^2) \ . \tag{3}$$

Recall that

$$S_0' = \pm p \ , \quad S_1 = -\frac{1}{4}\log p^2 \ . \tag{4}$$

Hence,

$$S_2' = \frac{1}{8p^3}(2pp'' - 3p'^2) \ , \tag{5}$$

or, using the relation $p = \sqrt{2m(E-V)}$,

$$S_2' = -\frac{1}{32\sqrt{2m}}\left(\frac{5V'^2}{(E-V)^{5/2}} + \frac{4V''}{(E-V)^{3/2}}\right) \ . \tag{6}$$

2. The LO WKB approximation is applicable if

$$\psi = e^{\frac{i}{\hbar}(S_0 + \frac{\hbar}{i}S_1)}(1 + o(1)) \ . \tag{7}$$

This means that all exponents containing the higher-order terms must be close to one,

$$|e^{i\hbar^{n-1}S_n}| = 1 + o(1) \ \Rightarrow \ |\hbar^{n-1}S_n| \ll 1 \ , \ n \geqslant 2 \ . \tag{8}$$

3. We notice that S_2' can be rewritten as

$$S_2' = \frac{1}{4}\frac{d}{dx}\left(\frac{p'}{p^2}\right) + \frac{1}{8}\frac{p'^2}{p^3} \ , \tag{9}$$

and, hence,

$$\begin{aligned} |\hbar S_2| &= \left|\frac{1}{4}\hbar\frac{p'}{p^2} + \frac{1}{8}\int^x \hbar\frac{p'^2}{p^3}dx\right| \\ &\leqslant \frac{1}{4}\left|\frac{\hbar p'}{p^2}\right| + \frac{1}{8}\int^x\left|\frac{\hbar p'^2}{p^3}\right|dx = \frac{1}{4}|\lambda\!\!\!{}^-{}'| + \frac{1}{8}\int^x\left|\frac{\lambda\!\!\!{}^-{}'^2}{\lambda\!\!\!{}^-}\right|dx \ . \end{aligned} \tag{10}$$

Thus, from $|\lambda\!\!\!{}^-{}'| \ll 1$ and $\int^x |\lambda\!\!\!{}^-{}'^2\lambda\!\!\!{}^-{}^{-1}|\,dx \ll 1$ it follows that $|\hbar S_2| \ll 1$.

4. Differentiating the expression for the momentum, we find

$$p' = \frac{-2mV'}{2p} = \frac{m}{p}F \ , \quad F = -V' \ . \tag{11}$$

Next, we notice that

$$\frac{\hbar}{p}F = \lambda F \sim \delta A \ , \tag{12}$$

where δA is a work done by the force F on the distance λ. Finally,

$$T_{kin} = \frac{p^2}{2m} \ , \tag{13}$$

and we have

$$|\lambda'| = \left|\hbar\frac{p'}{p^2}\right| = \left|\frac{\hbar}{p}\frac{pp'}{m}\frac{m}{p^2}\right| \sim \left|\lambda F\frac{2m}{p^2}\right| \sim \left|\frac{\delta A}{T_{kin}}\right| \ll 1 \ . \tag{14}$$

2. On accuracy of the LO WKB approximation

1. In the classically forbidden region, the exponentially decaying LO WKB wave function is given by (up to a constant multiplier)

$$\psi = \frac{1}{\sqrt{|p|}}e^{-\frac{1}{\hbar}\int_{x_0}^{x}|p|dx} \ . \tag{15}$$

For the potential

$$V(x) = V_0\sqrt{\frac{x}{x_0}} \ , \quad x > 0 \tag{16}$$

we have at $x > x_0$

$$|p| = \sqrt{2V_0\left(\sqrt{\frac{x}{x_0}} - 1\right)} \ . \tag{17}$$

Integrating $|p|$, we arrive at

$$\psi = \left(2V_0\left(\sqrt{\frac{x}{x_0}} - 1\right)\right)^{-1/4} \exp\left[-\frac{4\sqrt{2V_0}}{15\hbar}x_0\left(\sqrt{\frac{x}{x_0}} - 1\right)^{3/2}\left(3\sqrt{\frac{x}{x_0}} + 2\right)\right] \ . \tag{18}$$

In the limit $x \gg x_0$, this simplifies to

$$\psi \sim V_0^{-1/4}(x/x_0)^{-1/8}e^{-\frac{4\sqrt{2V_0}}{15\hbar}x_0^{-1/4}x^{5/4}} \ . \tag{19}$$

2. One requires the LO result (**??**) to be accurate to 1 percent in the region $x > 2x_0$. This means, in particular, that the modulus of the NLO correction $\hbar S_2$ must not exceed 10^{-2} in that region, that is

$$|\hbar S_2|_{x>2x_0} \leqslant 10^{-2} \ . \tag{20}$$

To find S_2, we substitute the expression for the momentum (**??**) into eq. (**??**) and integrate over x. The result is

$$S_2 = \frac{3\frac{x}{x_0} - 4\sqrt{\frac{x}{x_0}} + 6 - 3\sqrt{\sqrt{\frac{x}{x_0}} - 1}\left(\sqrt{\frac{x}{x_0}} - \frac{x}{x_0}\right)\tan^{-1}\left(\sqrt{\sqrt{\frac{x}{x_0}} - 1}\right)}{96x_0\sqrt{2V_0}\sqrt{\frac{x}{x_0}}\left(\sqrt{\frac{x}{x_0}} - 1\right)^{3/2}} . \quad (21)$$

This is a monotonically decreasing function of x/x_0, hence the condition (**??**) implies

$$|\hbar S_2|_{x=2x_0} \leqslant 10^{-2} . \quad (22)$$

We have

$$|\hbar S_2|_{x=2x_0} \approx 0.14\frac{\hbar}{\sqrt{V_0}x_0} , \quad (23)$$

so,

$$V_0 \gtrsim \frac{196\hbar^2}{x_0^2} . \quad (24)$$

3. On asymptotics of the potential in the WKB approximation

1. At first glance it may seem that the power-like decreasing potential cannot endanger the validity of the WKB approach, since in the classically forbidden region the WKB wave function decays much faster (exponentially fast). Let us see, however, that this is not so. We have to verify that

$$|\lambda'| \ll 1 , \quad |S_1| \gg |\hbar S_2| , \quad |\hbar S_2| \ll 1 , \quad x \to \infty . \quad (25)$$

Let the potential behave as

$$V(x) \sim x^{-n} , \quad n > 0 , \quad x \to \infty . \quad (26)$$

In the following reasoning it is important that the energy of the particle coincides with the asymptotics of the potential at infinity (in our case, zero). Then, for the momentum we have

$$p \sim x^{-n/2} . \quad (27)$$

We can now check if the conditions (**??**) hold in the limit $x \to \infty$. For example,

$$|\hbar S_2| \sim \hbar x^{n/2-1} \ll 1 \quad \Rightarrow \quad \frac{n}{2} - 1 < 0 \quad \Rightarrow \quad n < 2 . \quad (28)$$

Next,

$$|S_1| \sim \frac{n}{4}\log x \gg 1 \quad \text{for any} \quad n > 0 . \quad (29)$$

Finally,

$$|\lambda'| \sim \hbar x^{n/2-1} \ll 1 \quad \Rightarrow \quad n < 2 . \quad (30)$$

Note that the case $n = 2$ is special and cannot be resolved without additional information about the potential. Hence, our best estimate is $n < 2$.

2. From the above we see that if the potential decreases faster than x^{-2}, the WKB approach fails to describe the decaying wave function at arbitrary large x. We also observe that any asymptotics of the form

$$V(x) \sim x^{-k} \ , \quad k < 2 \tag{31}$$

passes our tests of validity. But what if the potential falls off faster than any of (**??**), but still slower than x^{-2} ? Long story short, anything can happen. As an example, consider the following behavior,

$$V(x) \sim \left(\frac{\log x}{x}\right)^2 \ . \tag{32}$$

Here we have

$$p \sim \frac{\log x}{x} \ , \quad p' \sim \frac{1}{x^2} - \frac{\log x}{x^2} \sim \frac{\log x}{x^2} \ , \quad p'' \sim \frac{\log x}{x^3} \ . \tag{33}$$

So, for example,

$$S_2' \sim \frac{1}{x \log x} \quad \Rightarrow \quad |\hbar S_2| \sim \hbar \log\log x \ . \tag{34}$$

This increases with x! Thus, the LO approximation is wrong in this case. Note in parentheses that the other two conditions in (**??**) are satisfied.