QUANTUM PHYSICS III

Solutions to Problem Set 6

26 October 2018

1. The one-turning point problem

In the classically forbidden region $x < x_0$, the modulus of the momentum $p = \sqrt{2m(E-V)}$ is estimated as

$$|p| \sim |(x-x_0)^{\frac{1}{2}}| = -i(x-x_0)^{\frac{1}{2}} \ . \tag{1}$$

Switching to the polar coordinates,

$$\mathrm{Re}\,(x-x_0) = r\cos\phi \ , \quad \mathrm{Im}\,(x-x_0) = r\sin\phi \ , \tag{2}$$

we rewrite (1) as

$$|p| \sim e^{i\frac{3\pi}{2}} R^{\frac{1}{2}} e^{i\frac{\pi}{2}} \ . \tag{3}$$

Now we continue this expression to the right side from the turning point, the classically allowed region $x > x_0$. This is done by changing the phase of the point at which p is evaluated from π to either 0 or 2π :

$$e^{i\frac{3\pi}{2}} R^{\frac{1}{2}} e^{i\frac{\pi}{2}} \quad \rightarrow \quad e^{i\frac{3\pi}{2}} R^{\frac{1}{2}} e^{0} = -i(x-x_0)^{\frac{1}{2}} \ , \tag{4}$$

and

$$e^{i\frac{3\pi}{2}} R^{\frac{1}{2}} e^{i\frac{\pi}{2}} \quad \rightarrow \quad e^{i\frac{3\pi}{2}} R^{\frac{1}{2}} e^{i\pi} = +i(x-x_0)^{\frac{1}{2}} \ . \tag{5}$$

So, bypassing the turning point clockwise (eq. (4)), amounts to replacing $|p| \rightarrow -ip$, and bypassing the counterclockwise (eq. (5)), we replace $|p| \rightarrow +ip$. Hence, continuation of the WKB wave function

$$\psi(x)_{x<x_0} = \frac{C}{2\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_x^{x_0} |p(x')|dx'} \tag{6}$$

gives us two terms,

$$\frac{C}{2\sqrt{p(x)}} e^{-\frac{i}{\hbar}\int_{x_0}^{x} p(x')dx' + \frac{\pi}{4}} \ , \qquad \frac{C}{2\sqrt{p(x)}} e^{\frac{i}{\hbar}\int_{x_0}^{x} p(x')dx' - \frac{\pi}{4}} \ , \tag{7}$$

summing which, we have

$$\psi(x)_{x>x_0} = \frac{C}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_0}^{x} p(x')dx' - \frac{\pi}{4}\right) \ . \tag{8}$$

2. Quantization rule in a half-space

1. Let $x = x_0 > 0$ be the turning point of the semiclassical wave function. Continuing from the classically forbidden region $x > x_0$, we have at $0 < x < x_0$,

$$\psi_{x<x_0}(x) = \frac{C}{\sqrt{p}} \cos\left(\frac{1}{\hbar}\int_x^{x_0} p dx - \frac{\pi}{4}\right) . \tag{9}$$

Since the potential is regular at all $x > 0$, the LO WKB approximation stays valid arbitrarily close to zero. Then, the boundary condition $\psi(0) = 0$ implies that $\psi_{x<x_0}$ must approach zero as $x \to 0+$. Hence, eq. (9) must admit the form

$$\psi_{x<x_0}(x) = \frac{C'}{\sqrt{p}} \cos\left(\frac{1}{\hbar}\int_0^{x} p dx - \frac{\pi}{2}\right) . \tag{10}$$

One can rewrite the argument of the last cosine as

$$-\left(\frac{1}{\hbar}\int_x^{x_0} p dx - \frac{1}{\hbar}\int_0^{x_0} p dx + \frac{\pi}{2}\right) . \tag{11}$$

Comparing this with eq. (9), we deduce the condition

$$-\frac{1}{\hbar}\int_0^{x_0} p dx + \frac{\pi}{2} = -\frac{\pi}{4} + \pi n , \tag{12}$$

or

$$\int_0^{x_0} p dx = \pi\hbar\left(n + \frac{3}{4}\right) , \quad n = 0, 1, 2, ... \tag{13}$$

2. Using the quantization condition (13), we have

$$\begin{aligned}\int_0^{x_0} \sqrt{2m(E - V_0 - kx)}\, dx =& \frac{\sqrt{2m}}{k}(E - V_0)^{3/2} \int_0^{x_0} \sqrt{1 - y}\, dy \\ =& \frac{2}{3}\frac{\sqrt{2m}}{k}(E - V_0)^{3/2} = \pi\hbar\left(n + \frac{3}{4}\right) .\end{aligned} \tag{14}$$

Hence,

$$E_n = V_0 + \left(\frac{3k}{2\sqrt{2m}}\pi\hbar\left(n + \frac{3}{4}\right)\right)^{2/3} , \tag{15}$$

where $m = m_c/2$ is a reduced mass of the two-quark system. Numerically,

$$E_n \approx 1.42 \cdot n^{2/3}\ GeV. \tag{16}$$

3. WKB spectrum of the Harmonic oscillator

1. Let $\pm x_0$ be the turning points of the WKB wave function in the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 \ . \tag{17}$$

We have to compute the integral

$$\begin{aligned}\int_{-x_0}^{x_0} p dx = \int_{-x_0}^{x_0} \sqrt{2m\left(E - \frac{1}{2}m\omega^2 x^2\right)} dx = \sqrt{2mE}\int_{-x_0}^{x_0} \sqrt{1 - \frac{x^2}{x_0^2}} dx \\ = 2\sqrt{2mE}\int_0^{x_0} \sqrt{1 - \frac{x^2}{x_0^2}} dx \ .\end{aligned} \tag{18}$$

At this point, it is convenient to change the variables,

$$x = x_0 \sin\phi \ , \quad dx = x_0 \cos\phi \, d\phi \ . \tag{19}$$

The integral becomes,

$$\begin{aligned}\int_{-x_0}^{x_0} p dx &= 2\sqrt{2mE}\int_0^{\pi/2} x_0 \cos\phi \sqrt{1 - \sin^2\phi}\, d\phi \\ &= 2\sqrt{2mE}x_0 \int_0^{\pi/2} \cos^2\phi \, d\phi \\ &= 2\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}\frac{\pi}{4} = \frac{\pi E}{\omega} \ .\end{aligned} \tag{20}$$

Thus,

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \ . \tag{21}$$

We conclude that the LO WKB approach gives us the *exact* energy levels of the harmonic oscillator. This somewhat surprising fact is not unique to the potential (17). Later we will consider one more example of an integrable system whose energy levels are captured exactly by the LO approximation.

2. Here it is convenient to write the LO WKB applicability condition, $|\lambda'| \ll 1$, in the form (see the lectures)

$$V'^2 \gg \frac{\hbar}{\sqrt{m}} V''^{3/2} \ . \tag{22}$$

Applying this to (17) gives

$$m^2\omega^4 x_0^2 \gg \hbar\omega^3 m \quad \Rightarrow \quad m\omega^2 x_0^2 \gg \hbar\omega \ . \tag{23}$$

In other words,

$$E \gg \hbar\omega \ , \tag{24}$$

or $n \gg 1$, as expected. As a general remark, we note that although the applicability condition urges us not to trust the results of the WKB approach at small n, even for them the answer can, in fact, be surprisingly accurate (or even exact).

4. WKB spectrum in power-like potential

1. For the potential $V(x) = V_0 \left|\frac{x}{x_0}\right|^\alpha$, the integral to be calculated is

$$
\begin{aligned}
\int_{-x_0}^{x_0} p\, dx = \sqrt{2m} \int_{-x_0}^{x_0} \sqrt{E - V_0 \left|\frac{x}{x_0}\right|^\alpha}\, dx = 2\sqrt{2mE} \int_0^{x_0} \sqrt{1 - \frac{V_0}{E}\left|\frac{x}{x_0}\right|^\alpha}\, dx \\
=2\sqrt{2m} V_0^{-\frac{1}{\alpha}} x_0 E^{\frac{1}{2}+\frac{1}{\alpha}} \int_0^1 \sqrt{1 - y^\alpha}\, dy \\
=a E^{\frac{1}{2}+\frac{1}{\alpha}} \ ,
\end{aligned}
\tag{25}
$$

where

$$
a = \sqrt{2m\pi} V_0^{-\frac{1}{\alpha}} x_0 \frac{\Gamma(1+\frac{1}{\alpha})}{\Gamma(\frac{3}{2}+\frac{1}{\alpha})} \ . \tag{26}
$$

Then, the Bohr-Sommerfeld quantization condition gives

$$
E_n = \left[\frac{\pi\hbar}{a}\left(n + \frac{1}{2}\right)\right]^{\frac{1}{\frac{1}{2}+\frac{1}{\alpha}}} \ . \tag{27}
$$

When $\alpha = 2$, a becomes equal $\pi\sqrt{\frac{m}{2}}\frac{x_0}{\sqrt{V_0}}$, and

$$
E_n = \frac{\pi\hbar}{\sqrt{2m}}\frac{\sqrt{V_0}}{x_0}\left(n + \frac{1}{2}\right) \ . \tag{28}
$$

One can express this through the frequency ω of the harmonic oscillator,

$$
\frac{\sqrt{V_0}}{x_0} = \sqrt{\frac{m}{2}}\omega \ , \tag{29}
$$

and we restore the expression (21).

2. From eq. (27) we see that at large n, the dependence of the energy of the bound state on its number becomes

$$
E \sim n^{\beta(\alpha)} \ , \quad \beta(\alpha) = \frac{1}{\frac{1}{2}+\frac{1}{\alpha}} \ . \tag{30}
$$

The function $\beta(\alpha)$ is shown on figure 1. Note first that the difference between the adjacent energy levels,

$$
\frac{E_n}{E_{n-1}} \sim \left(\frac{n}{n-1}\right)^\beta \approx 1 + \frac{\beta}{n} \ , \tag{31}
$$

goes to zero as α approaches zero. Hence, the flatter the potential, the denser the energy levels. In the opposite limit, $\alpha \to \infty$, we have $\beta \to 2$. This is fully expected once we notice that the limit of infinite α turns the potential into a box with two infinite walls located at $x = \pm x_0$ (see figure 2), and the energy levels of a particle in this box are proportional to n^2.

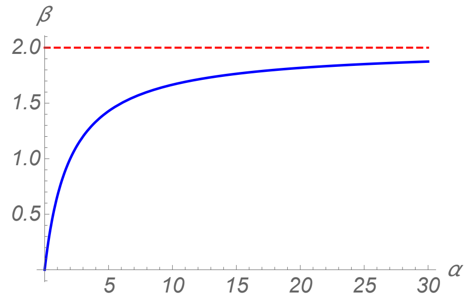

FIGURE 1 – The exponent of the energy of the bound state β plotted against the exponent of the potential α.

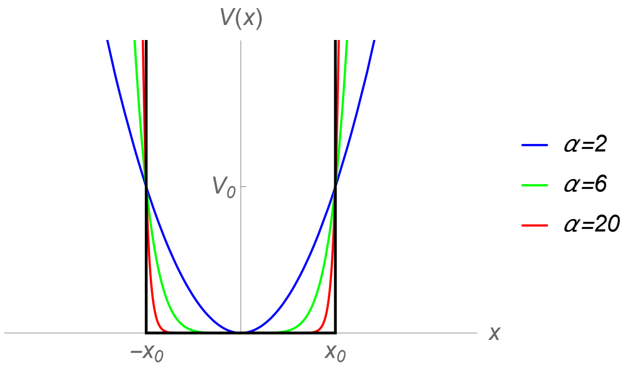

FIGURE 2 – The potentials $V = V_0|x/x_0|^\alpha$ for increasing values of α.

5. The multifold one-turning point problem

Here we use the method of bypassing the turning point in the complex plane. In the classically forbidden region $x > 0$, the dependence of the momentum p on the coordinate x is of the form

$$|p| \sim x^{k+\frac{1}{2}} \,. \tag{32}$$

We switch to the polar coordinates as in Problem 1 and rewrite the momentum as

$$|p| \sim R^{k+\frac{1}{2}} e^0 \,. \tag{33}$$

Now we continue this expression to the region $x < 0$ by changing the phase of the point from 0 to π or to $-\pi$. This gives

$$R^{k+\frac{1}{2}} e^0 \quad \to \quad R^{k+\frac{1}{2}} e^{i\pi k} e^{i\frac{\pi}{2}} \quad \text{or} \quad R^{k+\frac{1}{2}} e^0 \quad \to \quad R^{k+\frac{1}{2}} e^{-i\pi k} e^{-i\frac{\pi}{2}} \,. \tag{34}$$

Hence,

$$|p| \quad \to (-1)^k ip \quad \text{or} \quad |p| \quad \to \quad (-1)^{k+1} ip \,. \tag{35}$$

Applying these rules to the wave function

$$\psi(x)_{x>0} = \frac{C}{2\sqrt{|p|}} e^{-\frac{1}{\hbar}\left|\int_0^x p\, dx\right|} \,, \tag{36}$$

we get two contributions

$$\frac{C}{2\sqrt{p}} e^{\frac{i}{\hbar}\int_x^0 p\, dx \cdot (-1)^k - \frac{i\pi k}{2} - \frac{i\pi}{4}} \,, \qquad \frac{C}{2\sqrt{p}} e^{-\frac{i}{\hbar}\int_x^0 p\, dx \cdot (-1)^k + \frac{i\pi k}{2} + \frac{i\pi}{4}} \,, \tag{37}$$

and the answer is

$$\psi_{x<0}(x) = \frac{C}{\sqrt{p}} \cos\left(\frac{(-1)^k}{\hbar} \int_x^0 p\, dx - \frac{\pi}{4} - \frac{\pi k}{2} \right) . \tag{38}$$

Note that this expression matches with the general LO WKB solution in the allowed region. Note also that such matching is impossible unless k is an integer. The reason lies in the fact that if we promote the multiplicity of the turning point $2k + 1$ to an arbitrary real positive number, we loose analyticity around that point, and this makes the method used here inapplicable. In that case, it remains to solve the Schroedinger equation directly near the turning point as was done in Problem 5 of Problem set 5.

6*. Quantization rule beyond the LO

The solution will appear later.