QUANTUM PHYSICS III

Solutions to Problem Set 8

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1. Tunneling through a parabolic barrier

1. The transmission coefficient D is given by

$$D = e^{\frac{i\pi}{2}} e^{-\frac{1}{\hbar} \int_{-x^*}^{x^*} p\,dx} \;, \tag{1}$$

where $\pm x^*$ are the turning points. Computation of the tunneling exponent is straightforward :

$$
\begin{aligned}
\int_{-x^*}^{x^*} p\,dx &= \int_{-x^*}^{x^*} \sqrt{2m\left(V_0\left(1 - \frac{x^2}{x_0^2}\right) - E\right)}\,dx \\
&= x_0\sqrt{2mV_0} \int_a^b \sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}}\,\frac{dx}{x_0} \\
&= x_0\sqrt{2mV_0}\frac{1}{2}\left[\frac{x}{x_0}\sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}} + \left(1 - \frac{E}{V_0}\right)\mathrm{atan}\frac{x/x_0}{\sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}}} \right]_{x=-x^*}^{x=x^*} \\
&= x_0\sqrt{2mV_0}\frac{\pi}{2}\left(1 - \frac{E}{V_0}\right) \;,
\end{aligned}
\tag{2}
$$

where we used the relation $1 - (\pm x^*)^2/x_0^2 = E/V_0$. Finally,

$$D(E) = e^{\frac{i\pi}{2}} e^{-\frac{x_0}{2\hbar}(1 - E/V_0)\pi\sqrt{2mV_0}} \;. \tag{3}$$

2. The WKB approach reproduces well the value of the transmission coefficient provided that the tunneling exponent is large. Denoting $\delta E = V_0 - E$, we have from eq. (3),

$$\delta E \gg \frac{\hbar}{x_0}\sqrt{\frac{2V_0}{m}} \;. \tag{4}$$

2. Symmetry restoration in the double-well potential

1. Let $\psi_0(x)$ be the low energy bound state wave function of the left well. Then, one can build two eigenfunctions of the whole double-well potential as follows,

$$\psi_1(x) = \frac{1}{\sqrt{2}}(\psi_0(x) + \psi_0(-x)) , \quad \psi_2(x) = \frac{1}{\sqrt{2}}(\psi_0(x) - \psi_0(-x)) . \tag{5}$$

For the initial wave function of the particle in the left well we have

$$\Phi(x, 0) = \psi_0(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x)) . \tag{6}$$

Time evolving this wave packet, one finds

$$\Phi(x, t) = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}E_1 t}(\psi_1(x) + e^{-i\delta t}\psi_2(x)) , \quad \delta = \frac{E_2 - E_1}{\hbar} > 0 , \tag{7}$$

where δ is the energy splitting between the states represented by $\psi_1(x)$ and $\psi_2(x)$. The probability to detect the particle at the position x at the time t is then given by

$$P(x, t) = |\Phi\Phi^*| = \frac{1}{2}(\psi_1(x)^2 + \psi_2(x)^2 + 2\psi_1(x)\psi_2(x)\cos\delta t) . \tag{8}$$

2. The probability for the particle to be found in the right well at the time t can be written as

$$P(t) = \int_0^\infty P(x, t)\, dx . \tag{9}$$

We substitute eq. (8) into eq. (9), and note that integration of any term including the function $\psi_0(x)$ gives zero, since this function is localized in the left well, and that

$$\int_0^\infty \psi_0(-x)^2 dx = 1 , \tag{10}$$

because of the normalization of the bound state wave function. Hence,

$$P(t) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - 2\cos\delta t \cdot \frac{1}{2}\right) = \sin^2\frac{\delta t}{2} . \tag{11}$$

It remains to compute exactly the energy splitting δ,

$$\delta = \frac{\hbar\omega}{\pi}\exp\left(-\frac{1}{\hbar}\int_{-x^*}^{x^*} |p|dx\right) , \tag{12}$$

where $\pm x^*$ are the turning points of the subbarrier transition, and ω is the frequency of the classical oscillations in the well. The expression (12) can be easily computed in the limit $E \ll V_0$. Indeed, near the left well bottom the potential is well approximated by a parabolic function,

$$V(y) = V_0 y^2 (y + 2x_1)^2 \approx 4x_1^2 V_0 y^2 , \quad y = x - x_1 , \quad |y/x_1| \ll 1 . \tag{13}$$

From here, the oscillation frequency is extracted as

$$V(y) = \frac{1}{2}m\omega^2 y^2 \quad \Rightarrow \quad \omega^2 = \frac{8x_1^2 V_0}{m} \ . \tag{14}$$

Next, the integral in (12) is evaluated as follows,

$$\int_{-x^*}^{x^*} |p| dx \approx \sqrt{2mV_0} \int_{-x_1}^{x_1} \sqrt{(x-x_1)^2(x+x_1)^2} dx = \sqrt{2mV_0} \cdot \frac{4}{3}x_1^3 \ . \tag{15}$$

Thus,

$$\delta = \sqrt{\frac{8\hbar^2 x_1^2 V_0}{m\pi^2}} \exp\left(-\frac{4}{3\hbar} \sqrt{2mV_0} x_1^3\right) \ . \tag{16}$$

3. The average probability to detect the particle in the right well during the time T is given by

$$\frac{1}{T} \int_0^T P(t) \, dt \ . \tag{17}$$

Taking the limit $T \to \infty$, we obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_0^T (1 - \cos \delta t) \, dt = \frac{1}{2} - \frac{1}{2\delta} \lim_{T\to\infty} \frac{\sin \delta T}{T} = \frac{1}{2} \ . \tag{18}$$

So, indeed, we have the equal chance to find the particle in either well. In other words, in the large time limit, the system forgets its initial state and exhibits a universal behaviour. In particlar, the parity symmetry of the potential, broken by the initial distribution of the wave function, gets restored as the time passes by. Basically, this is the reason why in $1D$ quantum systems it is impossible to make a spontaneous symmetry breaking.

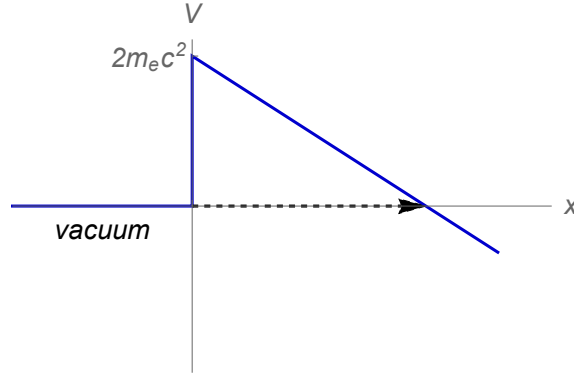3

3. Pair production in electric field

Fig. 1 – Effective potential

1. The effective potential of the problem is given by (see figure 1)

$$V(x) = 2m_e c^2 - |e|\mathcal{E}x, \quad x > 0, \quad V(x) = 0, \quad x \leqslant 0. \tag{19}$$

One should compute the probability to tunnel from the vacuum state with $E = 0$ to the state containing one electron-positron pair in the electric field. As the energy must be conserved, this state must also have zero energy. The tunneling probability is then given by

$$P(\mathcal{E}) = \exp\left(-\frac{2}{\hbar} \int_a^b |p| dx\right), \tag{20}$$

where $a = 0$ and $b = \frac{2m_e c^2}{|e|\mathcal{E}}$. Straightforward calculation gives

$$\int_a^b |p| dx = \int_a^b \sqrt{2m_e(2m_e c^2 - |e|\mathcal{E}x)} \, dx = \frac{8m_e^2 c^3}{3|e|\mathcal{E}}, \tag{21}$$

and

$$P(\mathcal{E}) = \exp\left(-\frac{16m_e^2 c^3}{3\hbar|e|\mathcal{E}}\right). \tag{22}$$

The answer is correct parametrically (i.e., up to the coefficient of the order of one in the exponent). The exact answer, obtained by the means of a more advanced technique of quantum field theory, yields a slightly different coefficient (π instead of 16/3) as well as some prefactor which cannot be caught in the LO WKB approximation.

The particle production becomes statistically significant when the argument of the exponent in eq. (22) approaches 1. Numerically, this implies the magnitude of the electric field

$$\mathcal{E} \sim 10^{12} \ V/cm. \tag{23}$$

This is a huge value. Note also that the argument can be rewritten as

$$\sim \frac{m_e c^2}{|e|\mathcal{E}\lambda_e}, \tag{24}$$

4

with λ_e the Compton wave length of the electron. This clarifies the physical meaning of the effect : the production of e^+e^- pairs becomes relevant, when the work done by the electric field ($|e|\mathcal{E}$) on the distance of the order of λ_e is comparable with the rest energy of the pair. Of course, the semiclassical approach is not applicable in this regime.

2. Strictly speaking, the e^+e^- pair production causes a back reaction on the electric field creating them. Indeed, the created particles move towards the sources of the field with the opposite charges, thus diminishing them. Although, when the effect is weak enough (as it is for all reasonable magnitudes of the electric field), this back reaction can be neglected, in an idealized situation, if one waits sufficiently long, one should be able to trace it. As an example of such situation, consider the flat vacuum capacitor with the parameters

$$Q = 10\,nC\,, \quad S = 10\,cm^2\,, \quad d = 1\,cm\,. \tag{25}$$

The initial electric field between the plates is given by

$$\mathcal{E}_0 = \frac{Q}{\epsilon_0 S}\,, \tag{26}$$

where ϵ_0 is the absolute permittivity. First, one should estimate the frequency of pair formation events in the volume $V = S d$ between the plates. We write the average time between two events as

$$\Delta t = \frac{\tau}{n}\,. \tag{27}$$

Here τ is the lifetime of the vacuum state at a given space position, and n is the number of elementary phase cells available in the volume V. The lifetime is estimated as follows (note that the particles are created in the rest),

$$\tau \sim \frac{2\pi}{\omega} P^{-1}(\mathcal{E})\,, \quad \omega = \frac{m_e c^2}{\hbar}\,, \tag{28}$$

where $P(\mathcal{E})$ is given in eq. (22). For n, we have

$$n \sim \frac{S d}{\lambda_e^3}\,, \quad \lambda_e = \frac{\hbar}{m_e c}\,. \tag{29}$$

Next, we note that the charge of the capacitor is decreased by $|e|$ at the time Δt. In other words,

$$\frac{\Delta \mathcal{E}}{\Delta t} = \frac{1}{S}\frac{\Delta Q}{\Delta t} = \frac{|e|n}{\epsilon_0 S \tau}\,. \tag{30}$$

We now replace the finite differences in this equation by differentials and use eqs. (22), (27)—(29) to obtain

$$\frac{d\mathcal{E}(t)}{dt} = B e^{-\frac{A}{\mathcal{E}(t)}}\,, \quad B = \frac{|e|d}{\epsilon_0}\frac{m_e^4 c^5}{\hbar^4}\,, \quad A = \frac{16}{3\hbar}\frac{m_e^2 c^3}{|e|}\,. \tag{31}$$

This must be supplemented with the initial condition

$$\mathcal{E}(0) = \mathcal{E}_0\,. \tag{32}$$

Eq. (31) is elementary integrated. The time T, at which the magnitude of the electric field is diminished by a factor 2, is given by

$$T = \frac{A}{B} \int_{\mathcal{E}_0/2A}^{\mathcal{E}_0/A} e^{\frac{1}{y}} dy \,. \tag{33}$$

Since the initial magnitude $\mathcal{E}_0 \sim 10^4$ V/cm is much smaller than (23), the limits of integration in (33) are very small, $\mathcal{E}_0/A \sim 10^{-8}$. At small y, the function $e^{\frac{1}{y}}$ is rapidly decreasing, and the value of the integral is evaluated at its lower limit. We arrive at the final answer

$$T \sim \frac{\mathcal{E}_0 \epsilon_0}{|e|d} \frac{\hbar^4}{m_e^4 c^5} P^{-1}(\mathcal{E}_0) \,. \tag{34}$$

4. Tunneling in a thermal bath

The quantum particle coupled to the classical thermal bath provides a good example of an interplay of classical and quantum effects. Here, classical thermal fluctuations push the particle to some excited state from which it then experiences quantum tunneling. On the one hand, it is more difficult to push the particle to a higher energy level than to a lower one. On the other hand, it is easier to tunnel from the higher energy level than from the lower one. Hence, at a given temperature, there exists the most probable energy level from which the particle escapes the well.

1. The probability $\tilde{P}(E)$ to escape the well from the level of energy E is a product of the probability to reach that level by a thermal fluctuation $e^{-\beta E}$, and the probability to tunnel through the barrier $P(E)$. Integrating over E, we write the full probability to escape the well as

$$P = \int_0^{E_{sph}} dE \, e^{-\beta E - B(E)} \,. \tag{35}$$

Now we apply the saddle-point approximation by saying that the integral above is saturated at the value of energy $E = E^*$, at which the expression in the exponent is minimized,

$$P \approx e^{-\beta E^* - B(E^*)} \,. \tag{36}$$

2. As the temperature increases, the thermal fluctuations become stronger and, eventually, the chance to jump over the barrier without tunneling becomes of the order of one. From this temperature, $E^* = E_{sph}$, hence $B(E^*) = 0$ and

$$P \approx e^{-\beta E_{sph}} \,. \tag{37}$$

3. One should find the minimum of the function

$$\beta E + B(E) \,, \tag{38}$$

6

where

$$B(E) = \frac{2}{\hbar} \int_{-x^*}^{x^*} \sqrt{2m(V - E)} \, dx$$

$$= \frac{2\sqrt{2m}}{\hbar} \int_{-x^*}^{x^*} \sqrt{E_{sph} - E_{sph} \left| \frac{x}{x_0} \right| - E} \, dx \tag{39}$$

$$= \frac{8\sqrt{2m}x_0(E_{sph} - E)^{3/2}}{3E_{sph}} \; .$$

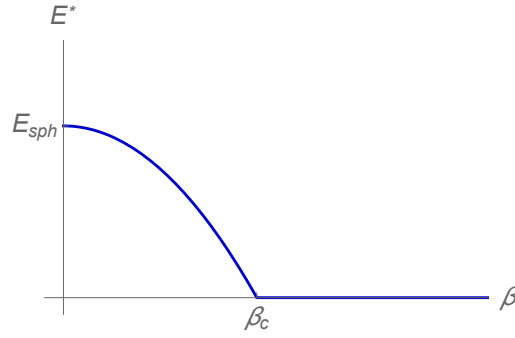

Fig. 2 – The most probable tunneling energy plotted against the inverse temperature.

Solving the equation

$$\frac{d}{dE}(\beta E + B(E)) \bigg|_{E=E^*} = 0 \; , \tag{40}$$

we find

$$E^* = \begin{cases} E_{sph} - \left(\dfrac{\beta E_{sph}}{4\sqrt{2m}x_0} \right)^2 \; , & \beta < \beta_c \\[2mm] 0 \; , & \beta \geqslant \beta_c \end{cases} \tag{41}$$

with

$$\beta_c = \frac{4\sqrt{2m}x_0}{E_{sph}} \; . \tag{42}$$

Thus, at the temperatures smaller than β_c^{-1}, the particle still prefers to tunnel from the ground state. At $T > \beta_c^{-1}$, the most probable energy rapidly increases and tends to E_{sph} in the limit $T \to \infty$ (see figure 2). Note that in this limit the semiclassical approximation breaks down, since the tunneling exponent becomes of the order of one, and the behavior of the particle is governed by the classical statistical physics.