QUANTUM PHYSICS III

Problem Set 9 17 November 2017

Reminders from classical scattering theory

1. Classical scattering on a Coulomb potential

Consider a constant flux of non-interacting particles (i.e. a constant number of n particles per area and time) of mass m with fixed energy and direction approaching a central potential $U(r)$ (a scattering center).

1. Show that the orbit equation for each individual particle is given by (see figure 1)

$$\phi(r) = \int_{\infty}^{r} \frac{L/r'^2 dr'}{\sqrt{2m(E - U(r')) - L^2/r'^2}} \,, \tag{1}$$

   with E the energy and L the angular momentum. For the Coulomb potential $U(r) = \alpha/r$ with $\alpha \in$ Reals and for $E > 0$ this is a scattering orbit (a hyperbola).

2. Use the previous equation to determine the deflection angle θ for a particle starting at $r = \infty$ and going back to $r = \infty$.

   Hint : Use the formula

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{x\sqrt{b^2 - 4ac}} \,. \tag{2}$$

3. Replace the constants of motion (E, L) by (E, b), with b the impact parameter, i.e. the normal distance between the asymptote of the incident particle and the scattering center at $r = 0$.

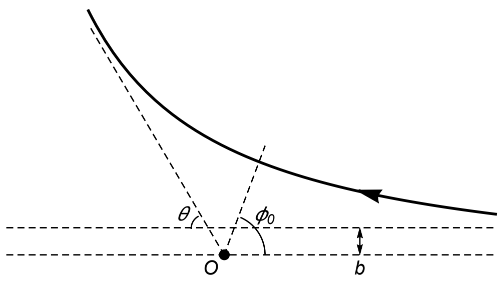

Fig. 1 – The Coulomb potential

4. Determine the number of particles dN per area and per time in a ring between b and $b + db$. If there is a one-to-one functional relation $b(\theta)$ between b and the scattering angle, then dN is at the same time the number of particles that is scattered in an

angle between θ and $\theta + d\theta$. Use this to show that the differential cross section for a Coulomb scattering (i.e. the Rutherford scattering formula) is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} \frac{1}{\sin^4 \frac{\theta}{2}} \ . \tag{3}$$

5. Show that the total cross section is infinite. Interpret the result.

2. Differential cross section transformation

Consider a particle of mass m_1 scattering off a target particle of mass m_2 in the non-relativistic limit.

1. Show that the relation between the differential cross section in the laboratory frame at a given lab angle θ_{LAB} and the differential cross section in the center of mass frame at the corresponding angle θ_{CM} can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{LAB}} = \frac{\left(1 + 2\lambda \cos\theta_{CM} + \lambda^2\right)^{3/2}}{|1 + \lambda \cos\theta_{CM}|} \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} \tag{4}$$

   with $\lambda = m_1/m_2$ the mass ratio of the two particles.

   Hint : Show that the relation between $\cos\theta_{\text{LAB}}$ and $\cos\theta_{\text{CM}}$ is given by

$$\cos\theta_{\text{LAB}} = \frac{\cos\theta_{\text{CM}} + \lambda}{(1 + 2\lambda\cos\theta_{\text{CM}} + \lambda^2)^{1/2}} \ . \tag{5}$$

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3. Interaction picture

Consider a system with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is the free Hamiltonian and $\hat{V}$ is the interaction. Define the interaction picture for states and operators via the relations

$$\begin{aligned} \Psi_I(t) &= \hat{U}_0^\dagger(t)\Psi_S(t) \ , \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t) \ , \end{aligned} \tag{6}$$

where $\hat{U}_0(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}$, and the subscript S denotes quantities in the Schrodinger picture.

1. Find the relation between the states and operators in the interaction and Heisenberg pictures.
2. Show that the evolution of the wave function in the interaction picture is described by the interaction term $\hat{V}$ in the same picture, i.e.

$$-\frac{\hbar}{i}\frac{d}{dt}\Psi_I(t) = \hat{V}_I\Psi_I(t) \, . \tag{7}$$

3. Express the evolution operator in the interaction picture $\hat{U}_I(t)$ through $\hat{U}(t)$ and $\hat{U}_0(t)$. Find a differential equation which $\hat{U}_I(t)$ obeys and determine the initial condition for it.

4. Unitarity versus isometry

Recall that the operator $\hat{U}$ acting in the Hilbert space $\mathcal{H}$ is called unitary if

$$\mathcal{D}(\hat{U}) = \mathcal{H} , \quad \mathcal{R}(\hat{U}) = \mathcal{H} , \quad \hat{U}^\dagger \hat{U} = 1 , \tag{8}$$

where the last equality should be understood in the operator sense,

$$\langle \Phi | \hat{U}^\dagger \hat{U} | \Phi \rangle = \langle \Phi | \Phi \rangle = 1 , \quad \forall \Phi \in \mathcal{H} . \tag{9}$$

1. Prove that the set of conditions (8) is equivalent to the following set,

$$\mathcal{D}(\hat{U}) = \mathcal{H} , \quad \hat{U}^\dagger \hat{U} = 1 , \quad \hat{U} \hat{U}^\dagger = 1 . \tag{10}$$

2. Prove that if $\mathcal{H}$ is finite-dimensional, the conditions (10) can be eased to

$$\mathcal{D}(\hat{U}) = \mathcal{H} , \quad \hat{U}^\dagger \hat{U} = 1 . \tag{11}$$

3. Assuming $\mathcal{H}$ to be infinite-dimensional and with the basis $|1\rangle, |2\rangle, ..., |n\rangle, ...$, construct the sequence of unitary operators $\hat{U}(\lambda)$ such that $\lim_{\lambda \to 0} \hat{U}(\lambda) = \hat{\Omega}$, where $\hat{\Omega}$ is an isometric non-unitary operator.