QUANTUM PHYSICS III

Solutions to Problem Set 9 16 November 2018

1. Classical scattering on a Coulomb potential

1. The energy of the particle moving in the potential $U(r)$ in two dimensions is written in polar coordinates as follows,

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + U(r)\,, \tag{1}$$

where $L = mr^2\dot{\phi}$ is the angular momentum. Expressing $\dot{r}$ from the relation above, we have

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}(E - U(r)) - \frac{L^2}{m^2r^2}}\,. \tag{2}$$

Next, we observe that as long as $\dot{r} \neq 0$,

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \frac{L}{mr^2}\frac{1}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{L^2}{m^2r^2}}}\,. \tag{3}$$

Taking the integral, we obtain

$$\phi(r) = \int_{\infty}^{r} \frac{L/r^2 dr}{\sqrt{2m(E - U(r)) - L^2/r^2}}\,. \tag{4}$$

2. The deflection angle θ is given by (see figure 1)

$$\theta = |\pi - 2\phi_0|\,, \tag{5}$$

where ϕ_0 is the angle between the direction to the minimum distance from the scattering center to particle's orbit and the direction to the infinite distance between them. Using eq. (4) with $U(r) = \alpha/r$, we write

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{Ldr}{r\sqrt{2mEr^2 - 2m\alpha r - L^2}}\,. \tag{6}$$

This integral can be taken analytically, the answer is

$$\phi_0 = \arcsin \left. \frac{-2m\alpha r - 2L^2}{r\sqrt{4m^2\alpha^2 + 8L^2mE}} \right|_{r_{min}}^{\infty}\,. \tag{7}$$

It remains to find r_{min}. It is the point at which $\dot{r} = 0$. From eq. (2) it then follows that

$$r_{min} = \frac{\alpha}{2E} + \frac{1}{2E}\sqrt{\alpha^2 + 2L^2E/m}\,. \tag{8}$$

Substituting this into eq. (7), we have

$$\phi_0 = -\arcsin \frac{\alpha}{\sqrt{\alpha^2 + 2L^2 E/m}} + \frac{\pi}{2} . \tag{9}$$

Hence,

$$\theta = 2 \arcsin \frac{|\alpha|}{\sqrt{\alpha^2 + 2L^2 E/m}} . \tag{10}$$

3. Since the energy and the angular momentum are conserved, one can write $E = mv_\infty^2/2$, hence $v_\infty = \sqrt{2E/m}$ and $L = mv_\infty b = \sqrt{2mE}b$. Eq.(10) is rewritten as

$$\theta = 2 \arcsin \frac{|\alpha|}{\sqrt{\alpha^2 + 4E^2 b^2}} . \tag{11}$$

4. Firstly, let us express b through θ :

$$b = -\frac{\alpha}{2E} \cot \frac{\theta}{2} . \tag{12}$$

Then, we note that $dN = 2\pi nbdb = 2\pi nb\frac{db}{d\theta}d\theta$, and $d\sigma = dN/n = 2\pi b\frac{db}{d\theta}d\theta$. On the other side, from eq. (12) we have

$$\frac{db}{d\theta} = -\frac{\alpha}{2E}\frac{1}{2}\frac{1}{\sin^2(\theta/2)} . \tag{13}$$

Hence,

$$d\sigma = 2\pi\frac{\alpha}{2E}\cot\left(\frac{\theta}{2}\right)\frac{\alpha}{2E}\frac{1}{2}\frac{d\theta}{\sin^2(\theta/2)} = \pi\frac{\alpha^2}{4E^2}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}d\theta . \tag{14}$$

Finally, $d\Omega = 2\pi \sin\theta d\theta = 4\pi \sin(\theta/2)\cos(\theta/2)d\theta$, and we arrive at

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\frac{1}{\sin^4(\theta/2)} . \tag{15}$$

5. The integral over eq. (15) is divergent. Therefore, the total cross section is infinite. The physical interpretation of this is that the potential affects the motion of the particle regardless its distance to the scattering center. This is a typical example of the so-called long-range force.

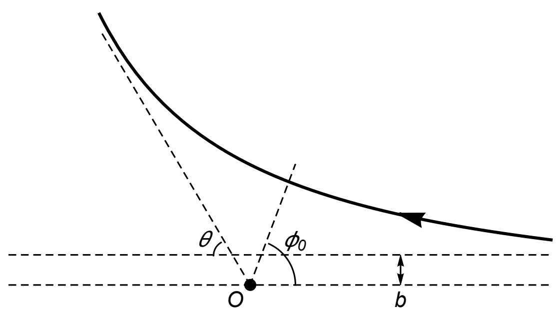

Fig. 1 – The scattering potential with $\alpha < 0$.

2. Differential cross section transformation

For the difference between the laboratory frame and the center-of-mass frame, see figure 2. In the center-of-mass frame, two particles are traveling towards each other. One particle with mass m_1 has a speed v_{C1} and is traveling in the $+x$-direction. The other particle with mass m_2 has a speed v_{C2} and is traveling in the $-x$-direction. Since we are in the center-of-mass frame, their momenta should be equal and opposite, so

$$v_{C2} = -\lambda v_{C1} \, , \tag{16}$$

where $\lambda = m_1/m_2$. After the collision, the first particle scatters into an angle θ_{CM} with the velocity u_{C1}. But $u_{C1} = v_{C1}$, because the collision is elastic. By the same reasoning, $u_{C2} = v_{C2}$. Expressing the velocities as vectors, we have

$$\begin{aligned} \vec{v}_{C1} &= v_{C1}\vec{i}\,, \\ \vec{u}_{C1} &= v_{C1}(\cos\theta_{CM}\vec{i} + \sin\theta_{CM}\vec{j})\,, \\ \vec{v}_{C2} &= -\lambda v_{C1}\vec{i}\,, \\ \vec{u}_{C2} &= \lambda v_{C1}(\cos\theta_{CM}\vec{i} + \sin\theta_{CM}\vec{j})\,. \end{aligned} \tag{17}$$

In the lab frame, the particle of mass m_1 comes in with the velocity v_{L1} and collides with the particle of mass m_2 with $v_{L2} = 0$ sending them both off in different directions. The scattered particle is deflected into an angle θ_{LAB} and has the velocity u_{L1}. The target particle is deflected into an angle θ_2 and has the velocity u_{L2}. As vectors, the velocities are

$$\begin{aligned} \vec{v}_{L1} &= v_{L1}\vec{i}\,, \\ \vec{u}_{L1} &= u_{L1}(\cos\theta_{LAB}\vec{i} + \sin\theta_{LAB}\vec{j})\,, \\ \vec{v}_{L2} &= 0\,, \\ \vec{u}_{L2} &= u_{L2}(\cos\theta_2\vec{i} - \sin\theta_2\vec{j})\,. \end{aligned} \tag{18}$$

To relate eqs. (17) and (18), we observe that the two reference frames are transformed to one another by a Galilean transformation, that is, to obtain the lab frame velocities, one should subtract v_{C2} in the x-direction from the center-of-mass frame velocities. Doing so, the velocities in the center-of-mass frame become,

$$\begin{aligned} \vec{v}_{L1} &= (1+\lambda)v_{C1}\vec{i}\,, \\ \vec{u}_{L1} &= v_{C1}((\cos\theta_{CM} + \lambda)\vec{i} + \sin\theta_{CM}\vec{j})\,, \\ \vec{v}_{L2} &= 0\,, \\ \vec{u}_{L2} &= -\lambda v_{C1}((\cos\theta_{CM} - 1)\vec{i} + \sin\theta_{CM}\vec{j})\,. \end{aligned} \tag{19}$$

Comparing eqs. (18) and (19), one finds

$$\begin{aligned} u_{L1}\cos\theta_{LAB} &= v_{C1}(\cos\theta_{CM} + \lambda)\,, \\ u_{L1}\sin\theta_{LAB} &= v_{C1}\sin\theta_{CM}\,. \end{aligned} \tag{20}$$

Hence,

$$\tan\theta_{LAB} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + \lambda}\,, \tag{21}$$

or

$$\cos\theta_{LAB} = \frac{1}{\sqrt{1+\tan^2\theta_{LAB}}} = \frac{\cos\theta_{CM}+\lambda}{\sqrt{1+2\lambda\cos\theta_{CM}+\lambda^2}}\,. \tag{22}$$

Since the total cross section should not depend on the reference frame, $d\sigma$ should be the same in either the lab frame or the center-of-mass frame. However, since there is an angular dependence in $d\Omega$, the differential cross section is different in different frames. Since

$$d\sigma = \frac{d\sigma}{d\Omega}d\Omega\,, \tag{23}$$

we know that

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} d\Omega_{CM} = \left.\frac{d\sigma}{d\Omega}\right|_{LAB} d\Omega_{LAB}\,. \tag{24}$$

Therefore,

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \left.\frac{d\sigma}{d\Omega}\right|_{LAB} \frac{d\Omega_{LAB}}{d\Omega_{CM}}\,. \tag{25}$$

Since $d\Omega = 2\pi\sin\theta d\theta$, we have,

$$\frac{d\Omega_{LAB}}{d\Omega_{CM}} = \frac{\sin\theta_{LAB}d\theta_{LAB}}{\sin\theta_{CM}d\theta_{CM}}\,. \tag{26}$$

Now, taking the derivative of eq. (22), we find

$$-\sin\theta_{LAB}d\theta_{LAB} = -\sin\theta_{CM}d\theta_{CM}\left(\frac{1+\lambda\cos\theta_{CM}}{(1+2\lambda\cos\theta_{CM}+\lambda^2)^{3/2}}\right)\,. \tag{27}$$

We then see by plugging this into eq. (26) that

$$\frac{d\Omega_{LAB}}{d\Omega_{CM}} = \frac{1+\lambda\cos\theta_{CM}}{(1+2\lambda\cos\theta_{CM}+\lambda^2)^{3/2}}\,, \tag{28}$$

and, finally,

$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \frac{(1+2\lambda\cos\theta_{CM}+\lambda^2)^{3/2}}{|1+\lambda\cos\theta_{CM}|}\left.\frac{d\sigma}{d\Omega}\right|_{CM}\,. \tag{29}$$

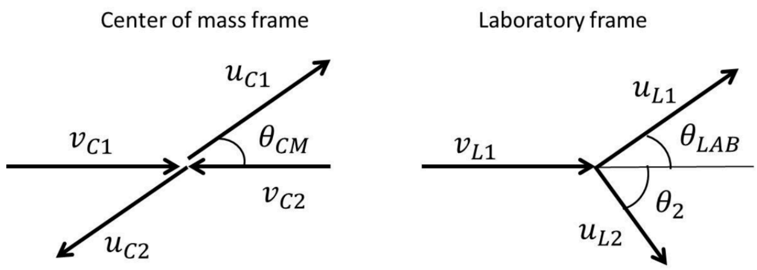

Fig. 2 – Different reference frames

3. Interaction picture

1. Recalling the relation between states and operators in the Schroedinger and Heisenberg pictures, we have

$$\begin{aligned} \Psi_I(t) &= \hat{U}_0^\dagger(t)\Psi_S(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\Psi_H \ , \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\hat{A}_H(t)\hat{U}^\dagger(t)\hat{U}_0(t) \ . \end{aligned} \tag{30}$$

2. The evolution equation for the wave function in the interaction picture is obtained straightforwardly :

$$\begin{aligned} -\frac{\hbar}{i}\frac{d}{dt}\Psi_I(t) &= -\frac{\hbar}{i}\frac{d}{dt}\hat{U}_0^\dagger(t)\Psi_S(t) = -\frac{\hbar}{i}\frac{d\hat{U}_0^\dagger(t)}{dt}\Psi_S(t) - \frac{\hbar}{i}\hat{U}_0^\dagger(t)\frac{d\Psi_S(t)}{dt} \\ &= -\hat{U}_0^\dagger(t)\hat{H}_0\Psi_S(t) + \hat{U}_0^\dagger(t)(\hat{H}_0 + \hat{V})\Psi_S(t) \\ &= \hat{U}_0^\dagger(t)\hat{V}\hat{U}_0(t)\Psi_I(t) = \hat{V}_I(t)\Psi_I(t) \ , \end{aligned} \tag{31}$$

where in the last line we used the fact that $\Psi_S(t) = \hat{U}_0(t)\Psi_I(t)$.

3. Similarly to the Schroedinger picture in which $\Psi_S(t) = \hat{U}(t)\Psi(0)$, one can define an operator $\hat{U}_I(t)$ such that $\Psi_I(t) = \hat{U}_I(t)\Psi(0)$. From eq. (30) we have

$$\Psi_I(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\Psi(0) \ . \tag{32}$$

Hence $\hat{U}_I(t) = \hat{U}_0^\dagger(t)\hat{U}(t)$. Substitution of eq. (32) into eq. (31) gives

$$-\frac{\hbar}{i}\frac{d\hat{U}_I(t)}{dt} = \hat{V}_I(t)\hat{U}_I(t) \ . \tag{33}$$

The initial condition for the operator $\hat{U}_I(t)$ is $\hat{U}_I(0) = 1$.

4. Unitarity versus isometry

1. (a) From $\mathcal{D}(\hat{U}) = \mathcal{H}$ and $\mathcal{R}(\hat{U}) = \mathcal{H}$ it follows that there is an inverse operator $\hat{U}^{-1}$ such that $\hat{U}\hat{U}^{-1} = 1$. Then, from $\hat{U}^\dagger\hat{U}\hat{U}^{-1} = \hat{U}^{-1}$ it follows that $\hat{U}^\dagger = \hat{U}^{-1}$. Therefore, $\hat{U}^{-1}\hat{U}\hat{U}^\dagger = \hat{U}^{-1}$, and $\hat{U}\hat{U}^\dagger = 1$.
(b) From $\hat{U}^\dagger\hat{U} = 1$ if follows that $\mathcal{R}(\hat{U}) \subseteq \mathcal{D}(\hat{U}^\dagger) = \mathcal{H}$. Then, from $\hat{U}\hat{U}^\dagger = 1$ it follows that for any element x from $\mathcal{D}(\hat{U}^\dagger)$ the operator $\hat{U}$ must map back to x the image of x under the action of $\hat{U}^\dagger$. Hence, $\mathcal{R}(\hat{U}) = \mathcal{D}(\hat{U}^\dagger) = \mathcal{H}$.

2. One should prove that if $\mathcal{H}$ is finite-dimensional, then $\mathcal{R}(\hat{U}) = \mathcal{H}$ follows from $\mathcal{D}(\hat{U}) = \mathcal{H}$. After that, $\hat{U}^\dagger\hat{U} = 1$ will follow from $\hat{U}\hat{U}^\dagger = 1$. To prove the coincidence of the domain and the range of $\hat{U}$, we enumerate the basis in $\mathcal{H}$ as $|1\rangle, ..., |n\rangle$. Then, $\hat{U}$ is represented by an $n \times n$ matrix. Since $\hat{U}^\dagger\hat{U} = 1$, it follows that $\det \hat{U} = 1$. Hence, $\hat{U}$ is non-degenerate and there is an inverse $n \times n$ matrix $\hat{U}^{-1}$. Thus, $\mathcal{D}(\hat{U}^{-1}) = \mathcal{H}$ and $\mathcal{R}(\hat{U}) = \mathcal{H}$.

3. To construct the required sequence, one can use the Gram–Schmidt orthogonalization process. Select the basis $|1\rangle, ..., |n\rangle, ...$ in $\mathcal{H}$. Choose the action of $\hat{U}(\lambda)$ on the vector $|1\rangle$ as follows,

$$\hat{U}(\lambda)|1\rangle = |1'\rangle = \sqrt{\lambda}\,|1\rangle + \sqrt{1-\lambda}\,|2\rangle\,. \tag{34}$$

We will consider λ in the range $[0,1]$. It is clear that $\langle 1'|1'\rangle = 1$. Now define the action of $\hat{U}(\lambda)$ on $|2\rangle$ as $\hat{U}(\lambda)|2\rangle = |2'\rangle = c_{21}|1\rangle + c_{22}|2\rangle + c_{23}|3\rangle$, and choose the coefficients c_{21}, c_{22}, c_{23} such that $\langle 1'|2'\rangle = 0$ and $\langle 2'|2'\rangle = 1$. The orthogonality condition fixes the values of c_{21} and c_{22},

$$c_{21} = -f(\lambda)\sqrt{1-\lambda}\,, \quad c_{22} = f(\lambda)\sqrt{\lambda}\,, \tag{35}$$

up to some arbitrary function $f(\lambda)$. One can choose, for example, $f(\lambda) = \sqrt{\lambda}$. Then, c_{23} is fixed by the normalization condition,

$$c_{23} = \sqrt{1-\lambda^2-\lambda(1-\lambda)}\,. \tag{36}$$

Hence

$$\hat{U}(\lambda)|2\rangle = |2'\rangle = -\sqrt{\lambda}\sqrt{1-\lambda}\,|1\rangle + \lambda\,|2\rangle + \sqrt{1-\lambda^2-\lambda(1-\lambda)}\,|3\rangle\,. \tag{37}$$

The next step of this procedure gives,

$$\hat{U}(\lambda)|3\rangle = |3'\rangle = c_{31}|1\rangle + c_{32}|2\rangle + c_{33}|3\rangle + c_{34}|4\rangle\,, \tag{38}$$

where

$$\begin{aligned}
c_{31} &= \sqrt{\lambda}(\lambda - \sqrt{1-\lambda^2-\lambda(1-\lambda)})\,,\\
c_{32} &= \sqrt{\lambda}(\sqrt{1-\lambda^2-\lambda(1-\lambda)} + \sqrt{\lambda}\sqrt{1-\lambda})\,,\\
c_{33} &= \sqrt{\lambda}(-\sqrt{\lambda}\sqrt{1-\lambda} - \lambda)\,,\\
c_{34} &= \sqrt{1-c_{31}-c_{32}-c_{33}}\,.
\end{aligned} \tag{39}$$

Since $\mathcal{H}$ is infinte-dimensional, one can continue this process and define the action of $\hat{U}(\lambda)$ on arbitrary $|n\rangle$. For all $\lambda \in (0,1]$, the operator $\hat{U}(\lambda)$ is unitary by construction. However, it is easy to see that in the limit of zero λ it becomes a "shift" operator

$$\hat{U}(0)|i\rangle \equiv \hat{\Omega}|i\rangle = |i+1\rangle, \quad \forall i, \tag{40}$$

whose range does not include the vector $|1\rangle$.