QUANTUM PHYSICS III

Solutions to Problem Set 10 23 November 2018

1. On integrals involving the delta-function

1. Consider the integral

$$I = \int_{-\infty}^{\infty} dx\, f(x)\delta(ax^2 + bx + c)\,. \tag{1}$$

Denote the argument of the delta-function by $g(x)$. There are several possibilities :

— If the equation $g(x) = 0$ has no real roots, then the argument of the delta-function is never zero, hence $I = 0$.

— Suppose that the equation $g(x) = 0$ has two different real roots $x_{1,2}$. Near each of them the function $g(x)$ can be written as $g(x) = g'(x_{1,2})(x - x_{1,2}) + O((x - x_{1,2})^2)$. Let O_1 and O_2 be small neighborhoods of the points x_1 and x_2 correspondingly. The integral I becomes

$$I = \int_{O_1} \frac{dy}{|g'(x_1 + y)|} f(x_1 + y)\delta(y) + \int_{O_2} \frac{dy}{|g'(x_2 + y)|} f(x_2 + y)\delta(y)\,, \tag{2}$$

where we made the change of variable $y = x - x_1$ in the first integral, $y = x - x_2$ in the second integral, and used the property of the delta-function

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)\,, \tag{3}$$

with α some constant. Taking the integrals, we have

$$I = \frac{f(x_1)}{|g'(x_1)|} + \frac{f(x_2)}{|g'(x_2)|}\,. \tag{4}$$

Finally, $|g'(x_1)| = |g'(x_2)| = |a(x_1 - x_2)| = \sqrt{b^2 - 4ac}$, and

$$I = (f(x_1) + f(x_2))(b^2 - 4ac)^{-1/2}\,. \tag{5}$$

— Suppose now that $x_1 = x_2 = x_0$. Expanding $g(x)$ around x_0 and changing the variable $y = x - x_0$, we arrive at

$$I = \int_{-\infty}^{\infty} \frac{dy f(x_0 + y)}{|g'(x_0 + y)|}\delta(y) = \lim_{x\to x_0} \frac{f(x)}{2|a(x - x_0)|} = \begin{cases} \infty\,, & f(x_0) \neq 0\,, \\ \dfrac{f'(x_0)}{2|a|}\,, & f(x_0) = 0\,. \end{cases} \tag{6}$$

2. Recall that
$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \,, \tag{7}$$
where i numerates the roots of the function f. In our case $E_p = \frac{p^2}{2m}$, and
$$\begin{aligned}\int d^3\mathbf{p}\ \delta(E_{p'}-E_p)f(\mathbf{p}) &= \int d\Omega dp p^2 \delta(E_p - E_{p'})f(\mathbf{p}) \\ &= \int d\Omega dp p^2 \delta(p-p')\frac{2m}{2p'}f(\mathbf{p}) \\ &= mp' \int d\Omega f(\mathbf{n}) \,,\end{aligned} \tag{8}$$
where
$$\mathbf{n} = |\mathbf{p}'|\frac{\mathbf{p}}{|\mathbf{p}|} \tag{9}$$
is a vector of modulus $|\mathbf{p}'|$ in the direction of $\mathbf{p}$.

2. Free particle's Green function in three dimensions

1. By definition,
$$\hat{G}_0(z) = \frac{1}{z-\hat{H}_0} \,. \tag{10}$$
This means that
$$\hat{G}_0(z)|\mathbf{p}\rangle = \frac{1}{z-E_p}|\mathbf{p}\rangle \,, \quad E_p = \frac{p^2}{2m} \,. \tag{11}$$
Therefore,
$$\langle\mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle = \int d^3\mathbf{p}\langle\mathbf{x}|\hat{G}_0(z)|\mathbf{p}\rangle\langle\mathbf{p}|\mathbf{x}'\rangle = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{z-E_p}. \tag{12}$$

2. Let us first compute the radial part of the integral :
$$\begin{aligned}\langle\mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle &= 2\pi\frac{1}{(2\pi)^3}\int_0^\infty dp p^2 \int_0^\pi d\theta \sin\theta \frac{e^{ip|\mathbf{x}-\mathbf{x}'|\cos\theta}}{z-E_p} \\ &= -\frac{1}{(2\pi)^2}\int_0^\infty dp \frac{p^2}{ip|\mathbf{x}-\mathbf{x}'|}\frac{1}{z-E_p}\left(e^{ip|\mathbf{x}-\mathbf{x}'|} - e^{-ip|\mathbf{x}-\mathbf{x}'|}\right) \\ &= \frac{im}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty} dp p \frac{e^{ip|\mathbf{x}-\mathbf{x}'|}}{2mz-p^2} \,.\end{aligned} \tag{13}$$
The resulting integral can be computed by the method of residues. To this end, we close the contour of integration in the plane of complex p as shown in figure 1. This does not change the value of the integral, since in the upper half-plane the integrand approaches zero exponentially fast when the radius of the semi-circle goes to infinity. The integrand has two poles at $p = \pm\sqrt{2mz}$. Recall that $z = E + i\epsilon$, $\epsilon > 0$, hence the pole contributing to the integral is the one at $p = +\sqrt{2mz}$. Thus,

$$
\begin{aligned}
\langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle &= \frac{im}{2\pi^2|\mathbf{x}-\mathbf{x}'|} 2\pi i \ \mathrm{res}_{p=\sqrt{2mz}} \frac{p e^{ip|\mathbf{x}-\mathbf{x}'|}}{(p-\sqrt{2mz})(p+\sqrt{2mz})} \\
&= -\frac{m}{2\pi}\frac{e^{i\sqrt{2mz}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \ .
\end{aligned} \tag{14}
$$

3. The formula (12) tells us that the Fourier transform of the function $G_0(z,\mathbf{x},\mathbf{x}') = \langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle$ is

$$
G_0(z,\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')\frac{1}{z-E_p} \ . \tag{15}
$$

This implies in particular the conservation of the free particle momentum. We now use the momentum representation of the Green function to yield

$$
\begin{aligned}
&\langle \mathbf{x}|(z-\hat{H}_0)\hat{G}_0(z)|\mathbf{x}'\rangle = \int d^3\mathbf{p} d^3\mathbf{p}' d^3\mathbf{p}'' \langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|z-\hat{H}_0|\mathbf{p}''\rangle\langle \mathbf{p}''|\hat{G}_0(z)|\mathbf{p}'\rangle\langle \mathbf{p}'|\mathbf{x}'\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3\mathbf{p} d^3\mathbf{p}' d^3\mathbf{p}'' e^{i\mathbf{p}\cdot\mathbf{x}}\delta(\mathbf{p}-\mathbf{p}'')(z-E_{p''})\delta(\mathbf{p}''-\mathbf{p}')\frac{1}{z-E_{p'}}e^{-i\mathbf{p}'\cdot\mathbf{x}'} \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p} d^3\mathbf{p}' e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{p}'\cdot\mathbf{x}'}\delta(\mathbf{p}-\mathbf{p}') \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} = \delta(\mathbf{x}-\mathbf{x}') \ .
\end{aligned} \tag{16}
$$

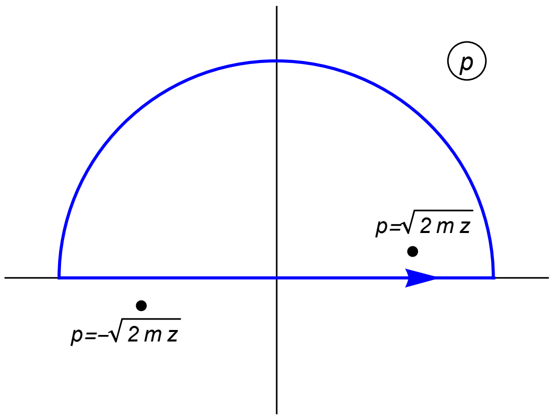

Fig. 1 – The contour of integration

4. The matrix element $G_0(z,\mathbf{x},\mathbf{x}')$ as a function of the complex variable z has a branch cut along the real positive values of z. To calculate the difference between the points on the opposite sides of the branch cut, one should continue analytically the function $\sqrt{z}$ from the one side to another. This gives,

$$
\begin{aligned}
G_0(E+i\epsilon,\mathbf{x},\mathbf{x}') - G_0(E-i\epsilon,\mathbf{x},\mathbf{x}') &= -\frac{m}{2\pi|\mathbf{x}-\mathbf{x}'|}\left(e^{i\sqrt{2mE}\,|\mathbf{x}-\mathbf{x}'|} - e^{-i\sqrt{2mE}\,|\mathbf{x}-\mathbf{x}'|}\right) \\
&= -\frac{im}{\pi|\mathbf{x}-\mathbf{x}'|}\sin\left(\sqrt{2mE}\,|\mathbf{x}-\mathbf{x}'|\right) \ .
\end{aligned} \tag{17}
$$

5. For large values of $x = |\mathbf{x}|$, the modulus $|\mathbf{x} - \mathbf{x}'|$ can be expanded as

$$|\mathbf{x} - \mathbf{x}'| = x\sqrt{1 - \frac{2\mathbf{x}\cdot\mathbf{x}'}{x^2} + \frac{x'^2}{x^2}} \approx x - \frac{\mathbf{x}\cdot\mathbf{x}'}{x} \ . \tag{18}$$

Thus,

$$G_0(z,\mathbf{x},\mathbf{x}') \approx -\frac{m}{2\pi}\frac{\exp\left[i\sqrt{2mz}\left(x - \frac{\mathbf{x}\cdot\mathbf{x}'}{x}\right)\right]}{x} \ , \quad x \to \infty \ . \tag{19}$$

Semiclassical S-matrix in one dimension

There is no easy way to do the integral with this transmission coefficient. You should replace that with $D(p) = 1 - e^{-p^2/p_0^2}$.

We want to compute the matrix element

$$S(\rho,\sigma,\rho',\sigma') = \int dxdy \langle\psi_{\rho'\sigma'}, x\rangle \langle x| S |y\rangle \langle y, \psi_{\rho\sigma}\rangle \tag{20}$$

By inserting the a complete set of momentum states we also know

$$\langle x| S |y\rangle = \int dqdq' \langle x, q\rangle \langle q| S \langle q', y\rangle \tag{21}$$

Finally the question says that we should consider ψ_{out} to just be the transmitted wave. So we find that $\langle q| S |q'\rangle = D(p)\delta(p - p)$. Here the delta function enforces that the wave is transmitted $p = p'$. (If we also consider the reflected wave part of ψ_{out} there would be another matrix element with $T(p)\ \delta p + p'$.

That leaves only an integral to be done (see the attached Mathematica file).