EPFL Automne 2018

Geometrie, MATH-120

Solutions Série 9

Solution 1. 1. To show that $\mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$ is a group. We first observe that the identity map $\mathrm{Id} : \mathrm{Id}(\mathbf{P}) = \mathbf{P}$ is in $\mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$. Also if $\phi_1, \phi_2 \in \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$, then so is the composition $\phi_1 \circ \phi_2$, since $(\phi_1 \circ \phi_2)(\mathbf{P}) = (\phi_1)(\phi_2(\mathbf{P})) = \phi_1(\mathbf{P}) = \mathbf{P}$. Finally, if $\phi \in \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$, that is $\phi(\mathbf{P}) = \mathbf{P}$, then $\phi^{-1}(\mathbf{P}) = \mathbf{P}$, so that the inverse ϕ^{-1} of ϕ is also inside $\mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$. Therefore $\mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$ is a group.

2. We claim that $\mathrm{Isom}(\mathbb{R}^2)_{\psi(\mathbf{P})} = \psi \circ \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}} \circ \psi^{-1}$. To see this, for any $f \in \mathrm{Isom}(\mathbb{R}^2)_{\psi(\mathbf{P})}$ (so that $f(\psi(\mathbf{P})) = \psi(\mathbf{P})$), one has $f = \psi \circ g \circ \psi^{-1} \in \psi \circ \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}} \circ \psi^{-1}$, where $g = \psi^{-1} f \psi \in \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$, because $g(\mathbf{P}) = \psi^{-1} f(\psi(\mathbf{P})) = \psi^{-1}(\psi(\mathbf{P})) = \mathbf{P}$. Conversely, for any element $h \in \psi \circ \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}} \circ \psi^{-1}$, we need to show $h \in \mathrm{Isom}(\mathbb{R}^2)_{\psi(\mathbf{P})}$. The assumption $h \in \psi \circ \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}} \circ \psi^{-1}$ implies that there exists $g \in \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}}$ such that $h = \psi \circ g \circ \psi^{-1}$. Then $h(\psi(\mathbf{P})) = (\psi \circ g \circ \psi^{-1})(\psi(\mathbf{P})) = (\psi \circ g)(\mathbf{P}) = \psi(\mathbf{P})$, so that indeed $h \in \mathrm{Isom}(\mathbb{R}^2)_{\psi(\mathbf{P})}$.

Solution. 4.

1. Let $t_1 = t_{(1,0)}$, the translation by the vector $(1, 0)$, and let $t_2 = t_{(0,1)}$, the translation by the vector $(0, 1)$. Then any translation $t_{\vec{v}} \in T(G) = T(\mathbb{R}^2) \cap G$ can in fact be written as $t_{\vec{v}} = mt_1 + nt_2 = t_{m(1,0)} + t_{n(0,1)} = t_{(m,n)}$, translation by $\vec{v} = (m, n) = m + ni$, for some $m, n \in \mathbb{Z}$. So that $T(G) \cong \mathbb{Z} + \mathbb{Z}i$, where $\mathbb{Z} + \mathbb{Z}i = \{m + ni : m \in \mathbb{Z}, n \in \mathbb{Z}\} \subset \mathbb{C}$.
2. Let $r = r_{i,0} : z \mapsto iz$ be a rotation. Then $r^2 : z \mapsto i^2 z$, $r^3 : z \mapsto i^3 z = -iz$, and $r^4 : z \mapsto i^4 z = z$ (which is the identity map Id). Let $s : z \mapsto \bar{z}$ be the symmetry (with respect to the x-axis), which is of order 2. Then $G_0 = \langle r, s\rangle = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ is a Dihedral group. We can write G_0 as the disjoint union of G_0^+ and G_0^- : $G_0 = G_0^+ \sqcup G_0^-$, where $G_0^+ = \langle r\rangle = \{e, r, r^2, r^3\}$ and $G_0^- = s\langle r\rangle = \{s, sr, sr^2, sr^3\}$. Now we can identify any element r^k in G_0^+ with a transformation of complex numbers : $r^k : z \mapsto i^k z$. That is, $r^k(z) = r_{i^k,0}(z)$, where $k \in \{0, 1, 2, 3\}$.
3. In part 2, we remarked that $G_0 - G_0^+ = G_0^- = s\langle r\rangle = \{s, sr, sr^2, sr^3\}$, where $r(z) = r_{i,0}(z) = iz$ is a rotation by a complex number i and $s(z) = \bar{z}$ is a symmetry. Any element sr^k of G_0, where $k \in \{0, 1, 2, 3\}$, can be written in terms of transformation of complex numbers : $(sr^k)(z) = s(i^k z) = \overline{i^k z} = i^{-k}\bar{z}$. That is, $sr^k(z) = s_{i^k,0}(z)$, for $k \in \{0, 1, 2, 3\}$.
4. First any $\phi \in G$ can be written as $\phi = t \circ \phi_0$, where $t = t_{\phi(\mathbf{0})} \in T(G)$ and $\phi_0 = t_{-\phi(\mathbf{0})} \circ \phi \in G_0$. Now we would like to show that the expression $\phi = t \circ \phi_0$

is unique. For this we first notice that $T(G) \cap G_0 = \{\mathrm{Id}\}$, since for any $t_{\vec{v}} \in T(G) \cap G_0$ we must have $t_{\vec{v}}(\mathbf{0}) = \mathbf{0} + \vec{v} = \mathbf{0}$, which implies that $\vec{v} = \mathbf{0}$, so that $t_{\vec{v}} = t_{\mathbf{0}}$ is the identity map Id. Now suppose we have $\phi = t \circ \phi_0 = t' \circ \phi_0'$, where $t, t' \in T(G)$ and $\phi_0, \phi_0' \in G_0$. Then $t'^{-1} \circ t = \phi_0' \circ \phi_0^{-1} \in T(G) \cap G_0 = \{\mathrm{Id}\}$, which implies $t = t'$ and $\phi_0 = \phi_0'$, showing that the decomposition $\phi = t \circ \phi_0$ is indeed unique.

To write element $\phi = t \circ \phi_0$ of G as transformations on complex numbers, we separate into two cases : either $\phi_0 \in G_0^+$, or $\phi_0 \in G_0^-$. By Part **1**, we know one can write any $t \in T(G)$ as $t = t_{m+ni}$, translation by $(m, n) = m + ni$ for some $m, n \in \mathbb{Z}$.

Case a : If $\phi_0 \in G_0^+ = \{e, r, r^2, r^3\}$, then $\phi_0 = r^k$, so that $t \circ \phi_0(z) = t_{m+ni} \circ r^k(z) = t_{m+ni}(i^k z) = i^k z + m + ni = r_{i^k, m+ni}(z)$, for some $k \in \{0, 1, 2, 3\}$.

Case b : If $\phi_0 \in G_0^- = \{s, sr, sr^2, sr^3\}$, then one can write $\phi_0 = sr^k$ for some $k \in \{0, 1, 2, 3\}$, and then $t \circ \phi_0(z) = t_{m+ni}(sr^k(z)) = t_{m+ni} \circ s(i^k z) = t_{m+ni}(\overline{i^k z}) = \overline{i^k z} + m + ni = s_{i^k, m+ni}(z)$.

5. Denote by $t_{\vec{u}}$ translation by the vector $\vec{u} = (1/4, 1/4) = 1/4 + 1/4i$. Recall Exercise 1 Q. 2 says that for any isometry ψ, one has $\mathrm{Isom}(\mathbb{R}^2)_{\psi(\mathbf{P})} = \psi \circ \mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}} \circ \psi^{-1}$. In particular, let $\mathbf{P} \subset \mathbb{R}^2$ be the "figure 2" in our question then $\mathrm{Isom}(\mathbb{R}^2)_{\mathbf{P}} = G$. By letting $\psi = t_{\vec{u}}$ we have $\mathrm{Isom}(\mathbb{R}^2)_{t_{\vec{u}}(\mathbf{P})} = t_{\vec{u}} \circ G \circ t_{\vec{u}}^{-1} = t_{\vec{u}} \circ G \circ t_{-\vec{u}}$.

   As in Part 4, we consider two cases, depending on whether $\phi_0 \in G_0^+$ or $\phi_0 \in G_0^-$ for $\phi = t \circ \phi_0 = t_{m+ni} \circ \phi_0 \in G$.

   Case (i). In the former case, any element of $\mathrm{Isom}(\mathbb{R}^2)_{t_{\vec{u}}(\mathbf{P})}$ can be identified with the transformation : $t_{\vec{u}} \circ (t_{m+ni} \circ r^k) \circ t_{-\vec{u}}(z) = (t_{1/4+1/4i} \circ t_{m+ni}) \circ r^k(z - 1/4 - 1/4i) = t_{m+1/4+(n+1/4)i}(i^k(z - 1/4 - 1/4i)) = i^k(z - 1/4 - 1/4i) + m + \frac{1}{4} + (n + \frac{1}{4})i = i^k z + m + \frac{1}{4} + (n + \frac{1}{4})i - i^k(\frac{1}{4} + \frac{1}{4}i) = r_{i^k, \nu}(z)$, where $\nu = (n + \frac{1}{4})i - i^k(\frac{1}{4} + \frac{1}{4}i)$, $k \in \{0, 1, 2, 3\}$ and $m, n \in \mathbb{Z}$.

   Case (ii). In the latter case, that is, if $\phi = t \circ \phi_0 = t_{m+ni} \circ \phi_0 \in G$ with $\phi_0 \in G_0^- = \{s, sr, sr^2, sr^3\}$, then any element of $\mathrm{Isom}(\mathbb{R}^2)_{t_{\vec{u}}(\mathbf{P})}$ can be identified with the transformation : $t_{\vec{u}} \circ (t_{m+ni} \circ sr^k) \circ t_{-\vec{u}}(z) = t_{1/4+1/4i} \circ t_{m+ni} \circ sr^k(z - 1/4 - 1/4i) = t_{m+1/4+(n+1/4)i} \circ s(i^k(z - 1/4 - 1/4i)) = t_{m+1/4+(n+1/4)i}(\overline{i^k(z - 1/4 - 1/4i)}) = \overline{i^k(z - 1/4 - 1/4i)} + m + \frac{1}{4} + (n + \frac{1}{4})i = i^{-k}\bar{z} + i^{-k}(-\frac{1}{4} + \frac{1}{4}i) + m + \frac{1}{4} + (n + \frac{1}{4})i = s_{i^k, \nu}(z)$, where $\nu = i^{-k}(-\frac{1}{4} + \frac{1}{4}i) + m + \frac{1}{4} + (n + \frac{1}{4})i$, $k \in \{0, 1, 2, 3\}$ and $m, n \in \mathbb{Z}$.