QUANTUM PHYSICS III

Solutions to Problem Set 12

8 December 2017

1. Scattering in a square-well potential

1. Substituting the potential

$$V(r) = \begin{cases} -V_0 \,, & r < R \,, \\ 0 \,, & r > R \end{cases} \tag{1}$$

into eq. (**??**), we have

$$f(\mathbf{p} \to \mathbf{p}') = \frac{2mV_0}{q} \int_0^R dr\; r \sin qr \;. \tag{2}$$

Integrating by parts and taking a square, we obtain

$$\frac{d\sigma}{d\Omega} = 4R^6 m^2 V_0^2 \frac{(\sin qR - qR\cos qR)^2}{(qR)^6} \;. \tag{3}$$

The plot of this differential cross section in the units qR is shown in figure 1.

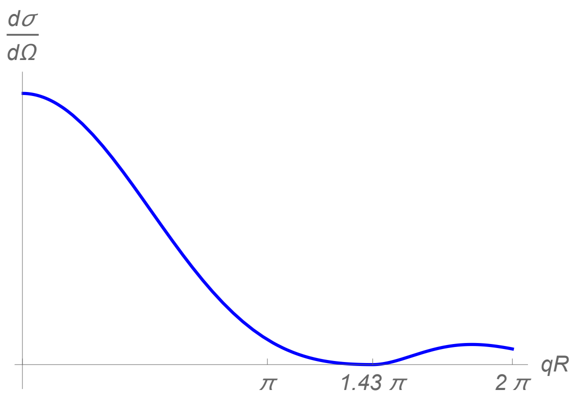

Fig. 1 – The plot of $d\sigma/d\Omega$ given by eq. (3) (not to scale).

2. Note that the distribution (3) develops a zero at the value of qR such that $qR = \tan qR$, i.e., at $qR \approx 1.43\pi$. Hence, by measuring the angle θ_* in which no scattering occurs, one can extract the value of R,

$$R \approx \frac{1.43\pi}{2p\sin\frac{\theta_*}{2}} \;. \tag{4}$$

3. In order that R may be found from the measuring of the zero point of the differential cross section (3), the maximum value of qR, $2pR$, must be larger than 1.43π, or

$$\begin{aligned} E \geqslant \frac{\hbar^2}{2m_p}\left(\frac{1.43\pi}{2R}\right)^2 &= \frac{(1.43\pi)^2}{8}\frac{\hbar^2}{m_p c^2}\left(\frac{c}{R}\right)^2 \\ &= \frac{(1.43\pi)^2}{8}\cdot\frac{(6.58\cdot 10^{-22})^2}{938}\cdot\left(\frac{3\cdot 10^{10}}{5\cdot 10^{-13}}\right)^2 \\ &= 4.2\ \mathrm{MeV}\ , \end{aligned} \tag{5}$$

where we restored $\hbar$ and c for numerical calculations.

4. From the formula for the momentum transfer, $q = 2p\sin\frac{\theta}{2}$, it follows that

$$d\Omega = d\phi\, d\cos\theta = 2\pi\frac{qdq}{p^2}\ . \tag{6}$$

Therefore,

$$\sigma = \int_0^{2p}\frac{d\sigma}{d\Omega}\frac{2\pi qdq}{p^2}\ . \tag{7}$$

Substituting eq. (3) we arrive after multiple integration by parts at

$$\sigma = \frac{2\pi}{p^2}(mV_0R^2)^2\left[1 - \frac{1}{(2pR)^2} + \frac{\sin 4pR}{(2pR)^3} - \frac{\sin^2 2pR}{(2pR)^4}\right]\ . \tag{8}$$

5. The slow scattering implies that the wave length $\lambda \sim p^{-1}$ of the scattered particles exceeds significantly the characteristic size of the potential. In our case this means $pR \ll 1$. Hence, to find the total cross section in this limit, we expand eq. (8) to the first nontrivial order in pR. This gives

$$\sigma = \frac{16\pi R^2}{9}(mV_0R^2)^2\ . \tag{9}$$

We observe that in the slow scattering regime the total cross section shows no dependence on the incident momentum of the particles. This is consistent with expectations, since the scattering amplitude (**??**) itself becomes independent of q in the limit $q \to 0$.

6. In the limit of fast scattering, $pR \gg 1$, we have

$$\sigma = \frac{2\pi}{p^2}(mV_0R^2)^2\ . \tag{10}$$

In agreement with expectations, the cross section goes to zero as the energy of the particles increases.

2. Towards the inverse scattering problem

1. Let R be the characteristic size of the potential $V(r)$. The scattering amplitude at zero momentum transfer f_0 is given by

$$f_0 \approx -\lim_{q\to 0} \frac{2m}{q} \int_0^R dr\, rV(r) \sin qr = -2m \int_0^R dr\, r^2 V(r)\ . \tag{11}$$

Next, we compute the amplitude assuming that $qR \ll 1$, this gives

$$\begin{aligned} f(q) &\approx -\frac{2m}{q} \int_0^R dr\, rV(r) \left(qr - \frac{1}{6}(qr)^3\right) \\ &= f_0 - f_0 \frac{(qR)^2}{10}\ . \end{aligned} \tag{12}$$

From here, one can extract the size R as

$$R^2 \approx \frac{10}{f_0} \frac{f_0 - f(q)}{q^2} \approx \frac{10}{f_0} \frac{|f'(q)|}{q} \approx \frac{10C}{f_0}\ . \tag{13}$$

2. Assume that at small distances the potential exhibits the power-like behavior, $V(r) \sim r^n$. Then,

$$f(q) \approx -\frac{2m}{q} \int_0^R dr\, r^{n+1} \sin qr = -\frac{2m}{q^{n+3}} \int_0^{qR} dy\, y^{n+1} \sin y\ , \tag{14}$$

where we denoted $y = qr$. If $qR \gg 1$, one can replace the upper limit of integration in the r.h.s. by infinity, hence at the large momentum transfers

$$f(q) \sim \frac{1}{q^{n+3}}\ . \tag{15}$$

Comparing this with the given data gives $\frac{N}{2} = n + 3$, and

$$V(r) \sim r^{\frac{N}{2}-3}\ , \quad r \to 0\ . \tag{16}$$

3. Truncation of the Coulomb potential

1. For the exponential shielding we find

$$\begin{aligned} f_1(\mathbf{p} \to \mathbf{p}') &= -\frac{2m}{q} \int_0^\infty dr\, r \frac{\alpha}{r} e^{-\frac{r}{\rho}} \sin qr \\ &= -\frac{2m\alpha}{2iq} \int_0^\infty dr \left(e^{r\left(iq-\frac{1}{\rho}\right)} - e^{r\left(iq-\frac{1}{\rho}\right)}\right) \\ &= -\frac{i\alpha m}{q} \left(\frac{1}{iq + \frac{1}{\rho}} + \frac{1}{iq - \frac{1}{\rho}}\right) = -\frac{2\alpha m}{q^2 + \rho^{-2}}\ . \end{aligned} \tag{17}$$

Note that the limit $\rho \to \infty$ is well-defined unless $q = 0$. Hence, except for the forward scattering cone, the amplitude for the exponentially truncated potential is ρ-independent for ρ large enough. In fact, in this limit f_1 reproduces the correct scattering amplitude for the Coulomb potential.

2. Evaluation of the sharp cutoff gives,

$$f_2(\mathbf{p} \to \mathbf{p}') = -\frac{2m}{q}\int_0^\rho dr\, r\frac{\alpha}{r}\sin qr = -\frac{2m\alpha}{q^2}(1 - \cos q\rho)\ . \tag{18}$$

This is again a well-defined expression but the one which has no limit at $\rho \to \infty$. We conclude that the answer for the scattering amplitude depends on *how* the truncation of the Coulomb potential is made. At first sight this fact seems distressing, but let us see how the truncation affects the quantities one can actually observe in experiment.

3. Let us assume that $q\rho \gg 1$, or

$$\rho \gg \frac{1}{2p\sin\frac{\theta}{2}}\ , \tag{19}$$

which can always be justified unless $\theta = 0$. Then, the ratio of the amplitudes (18) and (17) averaged over the range of the scattering angles from θ to $\theta + \Delta\theta$ is equal to

$$\frac{1}{\Delta\theta}\int_\theta^{\theta+\Delta\theta} d\theta' \left|\frac{f_2(\theta')}{f_1(\theta')}\right| = 1 - \frac{1}{\Delta\theta}\int_\theta^{\theta+\Delta\theta} d\theta' \cos\left(2p\rho\sin\frac{\theta}{2}\right)\ . \tag{20}$$

The integrand in the second term is a rapidly oscillating function that is integrated to zero provided that

$$2p\rho\left(\sin\frac{\theta+\Delta\theta}{2} - \sin\frac{\theta}{2}\right) \gg 2\pi\ . \tag{21}$$

For $\theta \neq \pi$, one can expand the sin to the first power in $\Delta\theta$ to obtain from eq. (21)

$$2p\rho\frac{1}{2}\frac{\Delta\theta}{2}\cos\frac{\theta}{2} \gg 2\pi \quad\Rightarrow\quad \rho \gg \rho_0 = \frac{4\pi}{p\Delta\theta\cos\frac{\theta}{2}}\ . \tag{22}$$

For $\theta = \pi$, we expand the sin to the second power in $\Delta\theta$ and find

$$2p\rho\frac{\Delta\theta^2}{8} \gg 2\pi \quad\Rightarrow\quad \rho \gg \rho_0 = \frac{8\pi}{p\Delta\theta^2}\ . \tag{23}$$

4. Given the in wave packet $\Psi_{in}(\mathbf{p})$, the out wave function $\Psi_{out}(\mathbf{p})$ is evaluated in the first Born approximation as

$$\begin{aligned}\Psi_{out}(\mathbf{p}) &= \Psi_{in}(\mathbf{p}) + \frac{i}{2\pi m}\int d^3\mathbf{p}'\delta(E_p - E_{p'})f(\mathbf{p} \to \mathbf{p}')\Psi_{in}(\mathbf{p}') \\ &= \Psi_{in}(\mathbf{p}) + \frac{ip}{2\pi}\int d\Omega_{p'}f(\mathbf{p} \to \mathbf{p}')\Psi_{in}(\mathbf{p}')\ .\end{aligned} \tag{24}$$

In this expression, the first term represents the unscattered incident wave, and to avoid seeing this term one normally restricts the measurements to non-forward directions. Then, the difference between the amplitudes f_1 and f_2 contains the rapidly oscillating term

$$\cos q\rho = \cos\left(2p\rho\sin\frac{\theta}{2}\right)\ ,\quad \theta \neq 0\ , \tag{25}$$

which, for ρ exceeding the size of the initial wave packet, integrates out to zero, and makes no contribution to $\Psi_{out}(\mathbf{p})$. Thus, the difference between the two methods of screening the Coulomb potential has no observable effect.

4*. The nucleus form factor

1. The potential $V(r)$ created by the charge distribution $\rho(r)$ satisfies Poisson's equation

$$\nabla^2 V(r) = \frac{1}{r}\frac{d^2}{dr^2}(rV(r)) = 4\pi e\rho(r) \ . \tag{26}$$

By eq. (**??**),

$$\begin{aligned} f(\mathbf{p} \to \mathbf{p}') &= -\frac{2m}{q}\int_0^\infty dr\ rV(r)\sin qr \\ &= \frac{2m}{q^2}\left[rV(r)\cos qr\right]_{r=0}^{r=\infty} - \frac{2m}{q^2}\int_0^\infty dr\ (rV(r))'\cos qr \\ &= \frac{2m}{q^2}\left[rV(r)\cos qr - \frac{1}{q}(rV(r))'\sin qr\right]_{r=0}^{r=\infty} + \frac{2m}{q^3}\int_0^\infty dr\ (rV(r))''\sin qr \ . \end{aligned} \tag{27}$$

It is clear that at large distances the real nucleus potential $V(r)$ becomes indistinguishable from the potential $V_0(r)$ created by the point nucleus. Therefore, there appears a problem of how to treat the scattering amplitudes computed for the Coulomb-like potential for which the standard scattering theory is inapplicable. In eq. (27), the problem is revealed by noticing that the boundary terms in the last line do not vanish. Instead of developing a new scattering theory, it takes much less efforts to *regularize* the potential, i.e., to assume that at very large distances $V(r)$ and $V_0(r)$ become falling off sufficiently fast to ensure the validness of the conventional scattering amplitudes. We do not discuss possible physical mechanisms of such suppression; in fact, we assume that it happens at the distances far beyond the scattering region we are interested in. For our results to make sense, one has to make sure that the physical observables are independent of a particular way of regularization (which is true), and that they are consistent with the results obtained within the rigorous approach (which is also true). Bearing the above in mind, we write

$$V(r)\ ,\ V_0(r) \sim \frac{Ze^2}{r}e^{-\alpha r}\ , \quad r \to \infty \ . \tag{28}$$

The original potentials are restored in the limit $\alpha \to 0$. In this regularization

$$f(\mathbf{p} \to \mathbf{p}') = \frac{8\pi m e}{q^3}\int_0^\infty dr\ r\rho(r)\sin qr \ , \tag{29}$$

while for the point-like nucleus

$$\begin{aligned} f_0(\mathbf{p} \to \mathbf{p}') &= \frac{2m}{q}\int_0^\infty dr\ Ze^2 e^{-\alpha r}\sin qr \\ &= \frac{2mZe^2}{q}\frac{q}{q^2+\alpha^2} \ . \end{aligned} \tag{30}$$

We observe that after the scattering amplitude is computed, one can safely remove the regularization by sending α to 0. Then, comparing eqs. (29) and (30), we obtain

$$F(\mathbf{q}^2) = \frac{4\pi}{Ze}\int_0^\infty dr\ r^2\rho(r)\frac{\sin qr}{qr} \ . \tag{31}$$

The generalization to the case of non-spherically symmetric charge distributions is therefore

$$F(\mathbf{q}^2) = \frac{1}{Ze} \int d\mathbf{x}\, \rho(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} \, . \tag{32}$$

That is, the form factor is the Fourier transform of the charge distribution.

2. Differentiating eq. (31) with respect to q, we have

$$\frac{dF}{dq} = \frac{4\pi}{Ze} \int_0^\infty dr\, r^2 \rho(r) \left[\frac{r \cos qr}{qr} - \frac{\sin qr}{q^2 r} \right] , \tag{33}$$

and

$$\frac{dF}{d(q^2)} = \frac{dF}{dq} \frac{dq}{d(q^2)} = \frac{1}{2q} \cdot \frac{dF}{dq} \, . \tag{34}$$

To find $dF/d(q^2)$ at $q^2 = 0$, we first compute

$$\begin{aligned} &\lim_{q\to 0} \left[\frac{r \cos qr}{qr} - \frac{\sin qr}{q^2 r} \right] \\ &\quad = \lim_{q\to 0} \left[\frac{r \cdot \left(1 - \frac{1}{2}(qr)^2\right)}{q^2 r} - \frac{qr - \frac{1}{6}(qr)^3}{q^3 r} \right] \\ &\quad = \lim_{q\to 0} \left(-\frac{r^2}{3} \right) = -\frac{r^2}{3} \, . \end{aligned} \tag{35}$$

Then

$$\left. \frac{dF}{d(q^2)} \right|_{q^2=0} = -\frac{1}{6} \cdot \frac{1}{Ze} \int_0^\infty dr\, r^2 \rho(r) \cdot 4\pi r^2 = -\frac{1}{6} \langle r^2 \rangle \, . \tag{36}$$

Thus, the mean-square radius of the proton is found from the experimental data as

$$\langle r^2 \rangle = -6 \left. \frac{dF}{d(q^2)} \right|_{q^2=0} \, . \tag{37}$$

Numerically

$$\sqrt{\langle r^2 \rangle} \approx 0.87 \cdot 10^{-13}\; cm \, . \tag{38}$$

This quantity is also called the charge radius of the proton.