QUANTUM PHYSICS III

Solutions to Problem Set 11 31 November 2018

1. Cauchy's theorem and the completeness relation

Let us compute directly the integral of $\hat{G}(z)$ over the contour C. The integral over the semi-circle gives

$$\begin{aligned}\int_{circle} \frac{dz}{z-\hat{H}} &= \lim_{R\to\infty} R \int_0^{2\pi} \frac{d\phi i e^{i\phi}}{Re^{i\phi}-\hat{H}} \\ &= \lim_{R\to\infty} \left[\mathrm{Log}\left(e^{i\phi} - \frac{\hat{H}}{R}\right)_{\phi=2\pi} - \mathrm{Log}\left(e^{i\phi} - \frac{\hat{H}}{R}\right)_{\phi=0} \right] = 2\pi i \ .\end{aligned} \tag{1}$$

The integral over the branch cut is computed as follows,

$$\begin{aligned}\int_{branch\ cut} \hat{G}(z) dz &= \int_0^\infty dx \int dp |p\rangle\langle p| \left(\frac{1}{x+i\epsilon - E_p} - \frac{1}{x - i\epsilon - E_p} \right) \\ &= -2\pi i \int dp |p\rangle\langle p| \ ,\end{aligned} \tag{2}$$

where $|p\rangle$ denote eigenstates of the continuous spectrum of $\hat{H}$, and we used the completeness of the eigenstates. Summing up the contributions (1) and (2), we obtain

$$\begin{aligned}\oint_C \hat{G}(z) dz &= 2\pi i \left(\sum_n |n\rangle\langle n| + \int dp |p\rangle\langle p| \right) - 2\pi i \int dp |p\rangle\langle p| \\ &= 2\pi i \sum_n |n\rangle\langle n| \ .\end{aligned} \tag{3}$$

This result tells us that the integral is given by the sum of the residues computed at the poles of $\hat{G}(z)$ located inside the contour. Hence, we recovered the Cauchy theorem.

2. Scattering amplitude in a spherically-symmetric potential

1. In the first Born approximation the scattering amplitude is written as

$$f(\mathbf{p} \to \mathbf{p}') = -\frac{m}{2\pi} \int d^3\mathbf{x} V(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} \; , \tag{4}$$

   where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ is the momentum transfer. For a spherically symmetric potential this expression reduces to

$$\begin{aligned} f(\mathbf{p} \to \mathbf{p}') &= -m \int_0^\infty dr\, r^2 V(r) \int_0^\pi d\theta\; \sin\theta e^{-iqr\cos\theta} \\ &= m \int_0^\infty dr\, r^2 V(r) \frac{e^{iqr} - e^{-iqr}}{-iqr} \\ &= -\frac{2m}{q} \int_0^\infty dr\, r \sin(qr) V(r) \; . \end{aligned} \tag{5}$$

2. With the potential

$$V(r) = V_0 e^{-r^2/a^2} \tag{6}$$

   the formula (5) becomes

$$\begin{aligned} f(\mathbf{p} \to \mathbf{p}') &= -\frac{2mV_0}{q} \int_0^\infty dr\, r e^{-r^2/a^2} \sin(qr) = -\frac{mV_0}{q} \int_{-\infty}^\infty dr\, r e^{-r^2/a^2} \sin(qr) \\ &= -\frac{mV_0 a^2}{2} \int_{-\infty}^\infty dr\, e^{-r^2/a^2} \cos(qr) = -\frac{mV_0 a^3}{2} \int_{-\infty}^\infty dr\, e^{-r^2} \cos(qar) \\ &= -\frac{mV_0 a^3}{4} \int_{-\infty}^\infty dr\, e^{-q^2a^2/4} \left(e^{-(r - iqa/2)^2} + e^{-(r + iqa/2)^2} \right) \\ &= -\frac{mV_0 a^3}{2} \sqrt{\pi} e^{-q^2a^2/4} \; , \end{aligned} \tag{7}$$

   where in going to the second line we integrated by parts. Hence,

$$\frac{d\sigma}{d\Omega} = \frac{\pi m^2 V_0^2 a^6}{4} e^{-q^2a^2/2} \; . \tag{8}$$

3. Applicability condition of the first Born approximation

1. First, we rewrite the difference $\Psi_{in}(\mathbf{x}) - \Psi_0(\mathbf{x})$ by using the relation between $\Psi_{in}(\mathbf{x})$ and $\Psi_0(\mathbf{x})$,

$$\begin{aligned} \Psi_0(\mathbf{x}) &= \Psi_{in}(\mathbf{x}) + \int d^3\mathbf{p} \langle \mathbf{x} | \hat{G}(E_p + i\epsilon) \hat{V} | \mathbf{p} \rangle \langle \mathbf{p} | \Psi_{in} \rangle \\ &= \Psi_{in}(\mathbf{x}) + \int d^3\mathbf{p} d^3\mathbf{x}' d^3\mathbf{x}'' \langle \mathbf{x} | \hat{G}(E_p + i\epsilon) | \mathbf{x}' \rangle \langle \mathbf{x}' | \hat{V} | \mathbf{x}'' \rangle \langle \mathbf{x}'' | \mathbf{p} \rangle \langle \mathbf{p} | \Psi_{in} \rangle \\ &= \Psi_{in}(\mathbf{x}) + \int d^3\mathbf{x}' G(E_p + i\epsilon, \mathbf{x}, \mathbf{x}') V(\mathbf{x}') \Psi_{in}(\mathbf{x}') \; . \end{aligned} \tag{9}$$

To the leading order in perturbation theory, in the last line one can replace G by G_0, since the difference between the free and the full Green functions provides the next-order correction to $\Psi_0(\mathbf{x})$. Since

$$\Psi_{in}(\mathbf{x}) \sim e^{i\mathbf{p}\cdot\mathbf{x}} , \tag{10}$$

the inequality $|\Psi_{in}(\mathbf{0}) - \Psi_0(\mathbf{0})| \ll |\Psi_{in}(\mathbf{0})|$ is rewritten as

$$\left| \int d^3\mathbf{x}' G_0(E_p - i\epsilon, \mathbf{0}, \mathbf{x}') V(\mathbf{x}') \Psi_{in}(\mathbf{x}') \right| \ll 1 . \tag{11}$$

Recall that

$$G_0(E_p + i\epsilon, \mathbf{x}, \mathbf{x}') = \frac{m}{2\pi} \frac{e^{i\sqrt{2mE_p}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} . \tag{12}$$

Then, if $V(\mathbf{x})$ contains only radial dependence on $\mathbf{x}$, one can perform explicitly the integration over angular variables in eq. (11) :

$$\begin{aligned} \left| \int d^3\mathbf{x}' G_0(E_p - i\epsilon, \mathbf{0}, \mathbf{x}') V(\mathbf{x}') \Psi_{in}(\mathbf{x}') \right| &= m \left| \int_0^\infty dr\, r^2 d\cos\theta \frac{e^{i\sqrt{2mE_p r}}}{r} V(r) e^{ipr\cos\theta} \right| \\ &= m \left| \int_0^\infty dr\, r e^{i\sqrt{2mE_p r}} V(r) \frac{1}{ipr} \left(e^{ipr} - e^{-ipr} \right) \right| \\ &\leqslant \frac{m}{p} \left| \int_0^\infty dr V(r) \left(1 - e^{2ipr} \right) \right| . \end{aligned} \tag{13}$$

Hence, eq. (11) takes the form

$$\frac{m}{p} \left| \int_0^\infty dr V(r) \left(1 - e^{2ipr} \right) \right| \ll 1 . \tag{14}$$

2. Consider the square well potential

$$V(r) = \begin{cases} -V_0 , & r < R , \\ 0 , & r > R . \end{cases} \tag{15}$$

In the slow scattering regime, $pR \ll 1$, we can expand the exponent in eq. (14) to the first order in pR to obtain

$$\frac{m}{p} \left| V_0 \int_0^R dr\, 2ipr \right| \sim mV_0R^2 \ll 1 . \tag{16}$$

3. Since $\sigma \sim 4\pi|f|^2$, and $f \sim \frac{m}{2\pi} V_0 \frac{4\pi}{3} R^3 \sim mV_0R^3$, we have by the virtue of eq. (16)

$$\sigma \sim 4\pi m^2 V_0^2 R^6 = 4\pi (mV_0R^2)^2 \cdot R^2 \ll 4\pi R^2 . \tag{17}$$

4. In the regime of fast scattering, $pR \gg 1$, the exponent in eq. (14) is a rapidly oscillating function which gives no overall contribution to the integral. Hence, the applicability condition becomes $\frac{m}{p} V_0 R \ll 1$, or

$$mV_0R^2 \ll pR . \tag{18}$$

Note that this requirement is much weaker than the condition (16) for slow particles. This is consistent with expectations, since for a given potential the Born approximation is supposed to work better as the energy of the scattered particles increases.

5. Applying the condition (14) to the Yukawa potential,

$$V(r) = \frac{\alpha}{r} e^{-\mu r}, \tag{19}$$

we have

$$\frac{m}{p} \left| \int_0^\infty dr \frac{\alpha}{r} e^{-\mu r} \sin pr \right| = \left| \frac{m\alpha}{p} \arctan \frac{p}{\mu} \right| \sim \begin{cases} \left| \frac{m\alpha}{\mu} \right| \ll 1, & p \ll \mu, \\ \left| \frac{m\alpha}{p} \right| \ll 1, & p \gg \mu. \end{cases} \tag{20}$$