QUANTUM PHYSICS III

Solutions to Problem Set 14 21 December 2018

1. Relativistic description of fermions

cf. section 14.3.

2. Non-relativistic limit of the Dirac equation

In this exercise we use the following notations :
$[\cdot,\cdot]$ denotes the vector product,
$[\cdot,\cdot]_-$ denotes the commutator,
$\{\cdot,\cdot\}_+$ denotes the anticommutator.

1. The Dirac equation

$$H\Psi = \mathcal{E}\Psi\ , \quad H = c\vec{\alpha}\vec{p} + \beta mc^2 + V\ , \quad \mathcal{E} = E + mc^2\ , \quad V = V(r)\ , \tag{1}$$

is rewritten in terms of the two-component spinors ϕ and χ as follows,

$$\begin{aligned} (E - V)\,\phi &= c\vec{\sigma}\vec{p}\,\chi\ , \\ (2mc^2 + E - V)\,\chi &= c\vec{\sigma}\vec{p}\,\phi\ . \end{aligned} \tag{2}$$

Using the second equation of this system, one can express χ through ϕ :

$$\begin{aligned} \chi &= \frac{1}{2mc^2}\frac{1}{1+\frac{E-V}{2mc^2}}c\vec{\sigma}\vec{p}\,\phi \\ &\approx \left(1 - \frac{E-V}{2mc^2}\right)\frac{\vec{\sigma}\vec{p}}{2mc}\phi + O(c^{-5})\ . \end{aligned} \tag{3}$$

Plugging this into the first equation of (2), we have, to the accuracy c^{-2},

$$(E - V)\,\phi = \frac{\vec{\sigma}\vec{p}}{2m}\left(1 - \frac{E-V}{2mc^2}\right)\vec{\sigma}\vec{p}\,\phi\ . \tag{4}$$

2. Let us rewrite eq. (4) in the form

$$\begin{aligned} E\phi + \frac{p^2}{2m}\frac{E}{2mc^2}\phi &= V\phi + \frac{p^2}{2m}\phi \\ &+ \frac{1}{2m}\frac{1}{2mc^2}(\vec{\sigma}\vec{p})V(\vec{\sigma}\vec{p})\phi \equiv \tilde{H}\phi\ . \end{aligned} \tag{5}$$

The operator $\tilde{H}$ is clearly Hermitian. The l.h.s. of the equation above is quite complicated as it contains the nontrivial differential operator p^2. To simplify the treatment, it is convenient to make the change of variable :

$$\xi = \sqrt{1 + \frac{p^2}{2m}\frac{1}{2mc^2}}\,\phi \equiv A\phi\ . \tag{6}$$

Then $\phi = A^{-1}\xi$ and

$$AE\xi = \tilde{H}A^{-1}\xi \ , \tag{7}$$

or

$$E\xi = A^{-1}\tilde{H}A^{-1}\xi \equiv H_{eff}\xi \ . \tag{8}$$

The operator H_{eff} is again Hermitian, and it can be thought of as an effective Hamiltonian of the quantum system whose state is represented by the spinor ξ. Eq. (8) is nothing but the stationary Schrodinger equation for this system.

3. First, we expand A^{-1} to the same accuracy as in eq. (4),

$$A^{-1} = 1 - \frac{1}{2}\frac{p^2}{2m}\frac{1}{2mc^2} + O(c^{-4}) \ . \tag{9}$$

Then, H_{eff} becomes

$$\begin{aligned} H_{eff} = &\left(1 - \frac{1}{2}\frac{p^2}{2m}\frac{1}{2mc^2}\right)\left[\frac{p^2}{2m} + V \right.\\ &\left. + \frac{1}{2m}\frac{1}{2mc^2}(\vec{\sigma}\vec{p})V(\vec{\sigma}\vec{p})\right]\left(1 - \frac{1}{2}\frac{p^2}{2m}\frac{1}{2mc^2}\right) \ . \end{aligned} \tag{10}$$

Extracting the term which does not depend on c, we obtain the leading-order equation on ξ,

$$\left(\frac{p^2}{2m} + V\right)\xi = E\xi \ . \tag{11}$$

As expected, this is the standard non-relativistic Schrodinger equation.

4. Now let us compute the leading relativistic corrections to eq. (11). The quadratic in c^{-1} term in H_{eff} reads

$$H^{(2)} = \frac{1}{2m}\frac{1}{2mc^2}(\vec{\sigma}\vec{p})V(\vec{\sigma}\vec{p}) - \frac{1}{2}\frac{1}{2mc^2}\left\{\frac{p^2}{2m}, \frac{p^2}{2m} + V\right\}_+ \ . \tag{12}$$

To bring this expression to the tractable form, we make use of the following operator identity,

$$(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{b}) = (\vec{a}\vec{b}) + i\vec{\sigma}\left[\vec{a}\ ,\vec{b}\right] \ , \tag{13}$$

valid for some vectorial operators $\vec{a}$ and $\vec{b}$. The product $(\vec{\sigma}\vec{p})V(\vec{\sigma}\vec{p})$ can therefore be transformed as

$$\begin{aligned} (\vec{\sigma}\vec{p})V(\vec{\sigma}\vec{p}) &= [\vec{\sigma}\vec{p}, V]_- \,\vec{\sigma}\vec{p} + V(\vec{\sigma}\vec{p})(\vec{\sigma}\vec{p}) \\ &= -i\hbar\left(\vec{\sigma}(\vec{\nabla}V)\right)\vec{\sigma}\vec{p} + Vp^2 \\ &= -i\hbar(\vec{\nabla}V)\vec{p} + \hbar\vec{\sigma}\left[\vec{\nabla}V, \vec{p}\right] + Vp^2 \ , \end{aligned} \tag{14}$$

where in going to the second line we used the fact that

$$[\vec{p}, f(x)]_- = -i\hbar\vec{\nabla}f(x) \ . \tag{15}$$

Now, we can write $H^{(2)}$ in the form

$$H^{(2)} = V_1 + V_2 + V_3 \ , \tag{16}$$

where

$$V_1 = -\frac{1}{2mc^2}\frac{(p^2)^2}{4m^2} \ , \tag{17}$$

$$V_2 = \frac{\hbar\vec{\sigma}}{4m^2c^2}\left[\vec{\nabla}V, \vec{p}\right] \ , \tag{18}$$

and

$$\begin{aligned} V_3 &= \frac{1}{2m}\frac{1}{2mc^2}(-i\hbar\vec{\nabla}V\vec{p}) + \frac{1}{2m}\frac{1}{2mc^2}Vp^2 - \frac{1}{2}\frac{1}{2m}\frac{1}{2mc^2}(p^2V + Vp^2) \\ &= \frac{1}{2m}\frac{1}{2mc^2}(-i\hbar\vec{\nabla}V\vec{p}) - \frac{1}{2}\frac{1}{2m}\frac{1}{2mc^2}[p^2, V]_- \ . \end{aligned} \tag{19}$$

Finally,

$$\begin{aligned} [p^2, V]_- &= p_ip_iV - Vp_ip_i = p_ip_iV - p_iVp_i + p_iVp_i - Vp_ip_i \\ &= p_i[p_i, V]_- + [p_i, V]_-p_i = p_i(-i\hbar)\nabla_iV + (-i\hbar\nabla_iV)p_i \ , \end{aligned} \tag{20}$$

and the two terms in eq. (19) are combined into

$$\begin{aligned} V_3 &= \frac{1}{2m}\frac{1}{2mc^2}(-i\hbar\vec{\nabla}V\vec{p}) - \frac{1}{2}\frac{1}{2m}\frac{1}{2mc^2}(p_i\nabla_iV + \nabla_iVp_i)(-i\hbar) \\ &- (-i\hbar)\frac{1}{2m}\frac{1}{2mc^2}\frac{1}{2}[p_i, \nabla_iV]_- \\ &= i\hbar\frac{1}{2m}\frac{1}{2mc^2}\frac{1}{2}(-i\hbar)\nabla^2V \\ &= \frac{\hbar^2}{8m^2c^2}\Delta V \ . \end{aligned} \tag{21}$$

5. Clearly, the first term (17) in H_{eff} represents the correction to the relativistic energy of the particle. To clarify the meaning of the other two, let us rewrite them for the particular case of an electron moving in the central field of a nucleus of charge Z. The potential of the problem is

$$V(r) = -\frac{Ze^2}{r} \ . \tag{22}$$

Therefore,

$$\vec{\nabla}V = \frac{Ze^2}{r^2}\frac{\vec{x}}{r} = \frac{\partial V}{\partial r}\frac{\vec{x}}{r} \ , \tag{23}$$

and

$$\Delta V = 4\pi\delta(\vec{x})\cdot Ze^2 \ . \tag{24}$$

Hence, the term V_3 becomes

$$V_3 = \frac{\hbar^2Ze^2\pi}{2m^2c^2}\delta(\vec{x}) \ . \tag{25}$$

Because of the presence of delta-function, it is called the contact (or Darwin) term. Next,

$$V_2 = \frac{\hbar\vec{\sigma}}{4m^2c^2}\frac{Ze^2}{r^3}[\vec{x}, \vec{p}] \ . \tag{26}$$

Here we recognize the orbital momentum operator and the spin operator :

$$\vec{L} = [\vec{x}, \vec{p}] , \quad \vec{s} = \hbar \frac{\vec{\sigma}}{2} . \tag{27}$$

So,

$$V_2 = \frac{1}{2m^2c^2} \frac{Ze^2}{r^3} \vec{s}\vec{L} . \tag{28}$$

Thus, V_2 represents the relativistic correction due to the spin-orbital interaction.

3. Zitterbewegung

1. The Heisenberg equation

$$i\hbar \dddot{x}_j = [\ddot{x}_j, H_D] \tag{29}$$

can be converted into the valid differential equation on the functions $\dot{x}_j(t)$ provided that we know how the latter commute with the Dirac Hamiltonian H_D. To find this, we write

$$i\hbar \ddot{x}_j = [\dot{x}_j, H_D] \tag{30}$$

and

$$i\hbar \dot{x}_j = [x_j, H_D] . \tag{31}$$

The last commutator is calculable straightforwardly : if

$$H_D = c\alpha_i p_i + \beta mc^2 , \tag{32}$$

then eq. (31) gives

$$i\hbar \dot{x}_j = c\alpha_i [x_j, p_i] = i\hbar c\alpha_j . \tag{33}$$

Next, we plug this result into eq. (30) and obtain

$$i\hbar \ddot{x}_j = c[\alpha_j, H_D] = mc^3[\alpha_j, \beta] = 2mc^3 \alpha_j \beta . \tag{34}$$

In obtaining the third equality, we kept only the leading term in c. We also used the Dirac representation of the Dirac matrices to compute the commutator $[\alpha_j, \beta]$. Finally,

$$i\hbar \dddot{x}_j = \frac{1}{i\hbar} 2m^2c^5 [\alpha_j \beta, \beta] = \frac{1}{i\hbar} 4m^2c^5 \alpha_j . \tag{35}$$

In the r.h.s. of this equation we recognize the expression for $\dot{x}_j$,

$$i\hbar \dddot{x}_j = \frac{1}{i\hbar} 4m^2c^4 \dot{x}_j . \tag{36}$$

Therefore, the differential equation on $\dot{x}_j(t)$ reads as follows,

$$\dddot{x}_j(t) + \frac{4m^2c^4}{\hbar^2} \dot{x}_j = 0 . \tag{37}$$

Its general solution is

$$\dot{x}_j(t) = a \sin \frac{2mc^2 t}{\hbar} + b \cos \frac{2mc^2 t}{\hbar} . \tag{38}$$

Integrating this, we obtain

$$x_j(t) = -\frac{\hbar}{2mc^2} a \cos \frac{2mc^2 t}{\hbar} + \frac{\hbar}{2mc^2} b \sin \frac{2mc^2 t}{\hbar} + d \; . \tag{39}$$

Here a, b and d are arbitrary operators.

2. Eqs. (33) and (34) are nothing but the relations the functions $x_j(t)$ must satisfy at $t = 0$, where the Schrodinger and the Heisenberg representations of operators coincide. Differentiating eq. (39) and comparing, we deduce

$$x_j(t) = x_j(0) + \frac{\hbar}{2mc} \left(\alpha_j \sin \frac{2mc^2 t}{\hbar} + i\alpha_j \beta \cos \frac{2mc^2 t}{\hbar} \right) \; . \tag{40}$$

3. Eq. (40) describes a rapidly oscillating trajectory, with the period $\hbar/2mc^2 = 6 \cdot 10^{-22}\; s$. The rapid motion of a "fermion at rest" is the Zitterbewegung, a peculiarity in the relativistic quantum mechanical motion of spin $1/2$ particle.
4. Because of this rapid motion of the Dirac particle, the net electric field the particle experiences is averaged over its "blur", and hence is somewhat different from the electric field at the position itself. The averaging of the electric field gives rise to the correction

$$\langle V \rangle = \frac{1}{2} \langle (\delta x^i)(\delta x^j) \rangle \frac{\partial^2 V}{\partial x^i \partial x^j} \; , \tag{41}$$

where $\delta x^i = x^i(t) - x^i(0)$. The isotropy tells us that

$$\langle (\delta x^i)(\delta x^j) \rangle = \frac{1}{3} \delta^{ij} \langle (\delta x^i)^2 \rangle = \delta^{ij} \frac{\hbar^2}{4m^2c^2} \; . \tag{42}$$

Then, the correction to the potential energy is

$$\langle V \rangle = \frac{1}{2} \frac{\hbar^2}{4m^2c^2} \Delta V \; , \tag{43}$$

and we have reproduced the Darwin term (21).