AN INTRODUCTION TO RIEMANNIAN GEOMETRY

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Contents

1. Introduction

Question 1.1. What kinds of quantities and operations appear in relation to analysis (or multivariable calculus) in a bounded open set $U \subset \mathbb{R}^n$?

Some possible answers:

- Functions: continuity, partial derivatives, integrals, L^p spaces, Taylor expansions, Fourier or related expansions
- Vector fields: gradient, curl, divergence
- Measures, distributions, flows
- Laplace operator, Laplace, heat and wave equations
- Integration by parts formulas (Gauss, divergence, Green)
- Tensor fields, differential forms
- Distance, distance-minimizing curves (line segments), area, volume, perimeter

Imagine similar concepts on a hypersurface (e.g. double torus in $\mathbb{R}^3$)

This course is an introduction to analysis on manifolds. The first part of the course title has the following Wikipedia description: "Mathematical Analysis is a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space)."

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Following this description, our purpose will be to study in particular differentiation, integration, and differential equations on spaces that are more general than the standard Euclidean space $\mathbb{R}^n$. Different classes of spaces allow for different kinds of analysis:

- *Topological spaces* are a good setting for studying continuous functions and limits, but in general they do not have enough structure to allow studying derivatives
- The smaller class of *metric spaces* admits certain notions of differentiability, but in particular higher order derivatives are not always well defined
- *Differentiable manifolds* are modeled after pieces of Euclidean space and allow differentiation and integration, but they do not have a canonical Laplace operator and thus the theory of differential equations is limited

The class of spaces studied in this course will be that of Riemannian manifolds. These are differentiable manifolds with an extra bit of structure, a Riemannian metric, that allows to measure lengths and angles of tangent vectors. Adding this extra structure leads to a very rich theory where many different parts of mathematics come together. We mention a few related aspects, and some of these will be covered during this course (the more advanced topics that will be covered will be chosen according to the interests of the audience):

(1) *Calculus.* Riemannian manifolds are differentiable manifolds, hence the usual notions of multivariable calculus on differentiable manifolds apply (derivatives, vector and tensor fields, integration of differential forms)
(2) *Metric geometry.* Riemannian manifolds are metric spaces: there is a natural distance function on any Riemannian manifold such that the corresponding metric space topology coincides with the usual topology. Distances are realized by certain distinguished curves called geodesics, and these can be studied via a second order ODE (the geodesic equation).
(3) *Measure theory.* Any oriented Riemannian manifold has a canonical measure given by the volume form. The presence of this measure allows to integrate functions and to define L^p spaces on Riemannian manifolds.
(4) *Differential equations.* There is a canonical Laplace operator on any Riemannian manifold, and all the classical linear partial differential equations (Laplace, heat, wave) have natural counterparts
(5) *Dynamical systems.* The geodesic flow on a closed Riemannian manifold is a Hamiltonian flow on the cotangent bundle, and the geometry of the manifold is reflected in properties of the flow (such as complete integrability or ergodicity)
(6) *Conformal geometry.* The notions of conformal and quasiconformal mappings make sense on Riemannian manifolds, and there is enough underlying structure to provide many tools for studying them
(7) *Topology.* There are several ways of describing topological properties of the underlying manifold in terms of analysis. In particular, Hodge theory characterizes the cohomology of the space via the Laplace operator acting on differential forms, and Morse theory describes the topological type of the space via critical points of a smooth function on it

(8) *Curvature.* The notion of curvature is fundamental in mathematics, and Riemannian manifolds are perhaps the most natural setting for studying curvature. Related concepts include the Riemann tensor, the Ricci tensor, and scalar curvature. There has been recent interest in lower bounds for Ricci curvature and their applications
(9) *Inverse problems.* Many interesting inverse problems have natural formulations on Riemannian manifolds, such as integral geometry problems where one tries to determine a function from its integrals over geodesics, or spectral rigidity problems where one tries to determine properties of the underlying space from knowledge of eigenvalues of the Laplacian.
(10) *Geometric analysis.* There are many branches of mathematics that are called geometric analysis. One particular topic is that of geometric evolution equations, where geometric quantities evolve according to a certain PDE. One of the most famous such equations is Ricci flow, where a Riemannian metric is deformed via its Ricci tensor. This was recently used by Perelman to complete Hamilton's program for proving the Poincaré and geometrization conjectures.

2. Calculus in Euclidean spaces

Let U be any nonempty open subset of R^n (not necessarily bounded, and could be equal to R^n). We fix standard Cartesian coordinates $x = (x_1, \cdots, x_n)$ and will use these coordinates throughout this chapter. We may sometimes write x^j instead of x_j, and we will also denote by v_j or v^j the j-th coordinate of a vector $v \in \mathbb{R}^n$.

2.1. **Functions and Taylor expansions.** Let $C(U)$ be the set of continuous functions on U. For partial derivatives, we will write

$$\partial_j f = \frac{\partial f}{\partial x_j} \quad \text{and} \quad \partial_{j_1 \cdots j_k} f = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}.$$

We denote by $C^k(U)$ the set of k times continuously differentiable real valued functions on U. Thus

$$C^k(U) = \Big\{ f\colon U \to \mathbb{R} : \partial_{j_1 \cdots j_k} f \in C(U) \text{ whenever } l \leq k \text{ and } j_1, \cdots, j_l \in \{1, \cdots, n\} \Big\}.$$

Recall also that if $f \in C^k(U)$, then $\partial_{j_1 \cdots j_k} f = \partial_{j_{\sigma(1)} \cdots j_{\sigma(k)}}$ for any permutation σ of $\{1, \cdots, k\}$.

We also denote by $C^\infty(U)$ the infinitely differentiable functions on U, that is,

$$C^\infty(U) = \bigcap_{k \geq 0} C^k(U).$$

Theorem 2.1 (Taylor expansion)**.** Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then

$$f(x) = \sum_{l=0}^{k} \frac{1}{l!} \Big[\sum_{j_1, \cdots, j_l} \partial_{j_1 \cdots j_l} f(x_0) (x - x_0)_{j_1} \cdots (x - x_0)_{j_l} \Big] + R_k(x; x_0),$$

where $|R_k(x; x_0)| \leq \eta(|x - x_0|) |x - x_0|^k$ for some function η with $\eta(s) \to 0$ as $s \to 0$.

Remark 2.2. The Taylor expansion of order 2 is given by

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}\nabla^2 f(x_0)(x - x_0)\cdots(x - x_0) + R_2(x; x_0),$$

where $\nabla f = (\partial_1 f, \cdots, \partial_n f)$ is the gradient of f and $\nabla^2 f(x) = \big(\partial_{jk} f(x)\big)_{j,k=1}^n$ is the Hessian matrix of f.

Proof. Considering $g(y) := f(x_0+y)$, we may assume that $x_0 = 0$. Assume that $B(0, r) \subset U$, fix $x \in B(0, r)$, and define

$$h\colon (-1-\varepsilon, 1+\varepsilon) \to U, \quad h(t) := g(tx),$$

where $\varepsilon > 0$ satisfies $(1 + \varepsilon)|x| < r$. Then h is a C^k function on $(-1 - \varepsilon, 1 + \varepsilon)$, and repeated use of the fundamental theorem of calculus gives

$$\begin{aligned}
h(t) &= h(t) - h(0) + h(0) = h(0) + \int_0^t h'(s)ds \\
&= h(0) + h'(0)t + \int_0^t (h'(s) - h'(0))ds = h(0) + h'(0)t + \int_0^t \int_0^s h''(u)duds \\
&= h(0) + h'(0)t + h''(0)\frac{t^2}{2} + \int_0^t \int_0^s (h''(u) - h''(0))duds \\
&= \cdots \\
&= \sum_{i=0}^k h^{(i)}(0)\frac{t^i}{i!} + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \big(h^{(k)}(t_k) - h^{(k)}(0)\big)dt_k \cdots dt_1. \qquad (2.1)
\end{aligned}$$

Here we used that $\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} dt_k \cdots dt_1 = \frac{t^k}{k!}$ (exercise).

Now, computation shows

$$h'(t) = \partial_j f(tx)x_j, \quad h''(t) = \partial_{jl} f(tx)x_j x_l, \quad \cdots$$

and

$$h^{(k)}(t) = \partial_{j_1\cdots j_k} f(tx)x_{j_1}\cdots x_{j_k}.$$

Applying (2.1) with $t = 1$ gives the result in the theorem, where

$$R_k(x) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \Big[\partial_{j_1\cdots j_k} f(t_k x) - \partial_{j_1\cdots j_k} f(0)\Big]x_{j_1}\cdots x_{j_k} dt_k \cdots dt_1.$$

The bound for R_k follows since $\partial_{j_1\cdots j_k} f$ is uniformly continuous on compact sets. □

At this point it may be good to mention another convenient form of the Taylor expansion, which we state but will not use. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers. Then $\mathbb{N}^n$ consists of all n-tuples $\alpha = (\alpha_1, \cdots, \alpha_n)$ where the α_j are nonnegative integers. Such an n-tuple is called a multi-index. We write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. For partial derivatives, the notation

$$\partial^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}$$

will be used. We also use the notation $\alpha! = \alpha_1!\cdots\alpha_n!$.

Theorem 2.3 (Taylor expansion, multi-index version)**.** Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then

$$f(x) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_k(x_0; x),$$

where R_k satisfies similar bounds as before.

Proof. Exercise. □

2.2. **Tensor fields.** If $f \in C^k(U)$, if $x \in U$ and if $v \in \mathbb{R}^n$ is such that $|v|$ is sufficiently small, we write the Taylor expansion given in Theorem 2.1 in the form

$$f(x + v) = \sum_{l=0}^{k} \frac{1}{l!} \Big[\sum_{j_1, \cdots, j_l = 1}^{n} \partial_{j_1 \cdots j_l} f(x) v_{j_1} \Big] + R_k(x + v; x).$$

The first few terms are

$$f(x + v) = f(x) + \partial_j f(x) v_j + \frac{1}{2} \partial_{jk} f(x) v_j v_k + \cdots$$

Looking at the terms of various degree motivates the following definition.

Definition 2.4 (Tensor fields)**.** An *m-tensor field* in U is a collection of functions $u = \big(u_{j_1 \cdots j_m}\big)_{j_1, \cdots, j_m = 1}^{n}$, where each $u_{j_1 \cdots j_m}$ is in $C^\infty(U)$. The tensor field u is called *symmetric* if $u_{j_1 \cdots j_m} = u_{j_{\sigma(1)} \cdots j_{\sigma(m)}}$ for any $j_1, \cdots, j_m$ and for any σ which is a permutation of $\{1, \cdots, m\}$.

Remark 2.5. This definition is specific to R^n, since we are deliberately not allowing any other coordinate systems than the Cartesian one. Later on we will consider tensor fields on manifolds, and their transformation rules under coordinate changes will be an important feature (these will decide whether the tensor field is covariant, contravariant or mixed). However, upon fixing a local coordinate system, all tensor fields will look essentially like the ones defined above.

Example 2.6.
(1) The 0-tensor fields in U are just the scalar functions $u \in C^\infty(U)$
(2) The 1-tensor fields in U are of the form $u = (u_j)_{j=1}^n$, where $u_j \in C^\infty(U)$. Thus 1-tensor fields are exactly the vector fields in U; the tensor $(u_j)_{j=1}^n$ is identified with $(u_1, \cdots, u_n)$.
(3) The 2-tensor fields in U are of the form $u = \big(u_{j,k}\big)_{j,k=1}^n$, where $u_{jk} \in C^\infty(U)$. Thus 2-tensor fields can be identified with smooth matrix functions in U. The 2-tensor field is symmetric if the matrix is symmetric.
(4) If $f \in C^\infty(U)$, then we have for any $m \geq 0$ an m-tensor field $u = \big(\partial_{j_1 \cdots j_m} f\big)_{j_1, \cdots, j_m = 1}^n$ consisting of partial derivatives of f. This tensor field is symmetric since the mixed partial derivatives can be taken in any order.

Again by looking at the terms in the Taylor expansion, one can also think that an m-tensor $u = \big(u_{j_1 \cdots j_m}\big)_{j_1, \cdots, j_m = 1}^n$ acts on a vector $v \in \mathbb{R}^n$ by the formula

$$v \mapsto u_{j_1 \cdots j_m}(x) v^{j_1} \cdots v^{j_m}.$$

The last expression can be interpreted as a multilinear map acting on the m-tuple of vectors $(v, \cdots, v)$.

Definition 2.7 (Multi-linear map)**.** If $m \geq 0$, an m-linear map is any map

$$L\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$$

such that L is linear in each of its variables separately.

The following theorem is almost trivial, but for later purposes it will be good to know that a tensor field can be thought of in two ways: either as a collection of coordinate functions, or as a map on U that takes values in the set of multilinear maps.

Theorem 2.8 (Tensors as multilinear maps)**.** If $u = \big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^n$ is an m-tensor field on $U \subset \mathbb{R}^n$, then for any $x \in U$, there is an m-linear map $u(x)$ defined via

$$u(x)(v_1, \cdots, v_m) = u_{j_1\cdots j_m}(x) v_1^{j_1} \cdots v_m^{j_m}, \quad v_1, \cdots, v_m \in \mathbb{R}^n,$$

and it holds that $u_{j_1\cdots j_m}(x) = u(x)(e_{j_1}, \cdots, e_{j_m})$. Conversely, if T is a function that assigns to each $x \in U$ an m-linear map $T(x)$, and if the function $u_{j_1\cdots j_m}\colon x \mapsto T(x)(e_{j_1}, \cdots, e_{j_m})$ are in $C^\infty(U)$ for each $j_1, \cdots, j_m$, then $(u_{j_1\cdots j_m})$ is an m-tensor field in U.

Proof. Exercise. □

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