Exercise One

1. Show that $\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} dt_k \cdots dt_1 = \frac{t^k}{k!}$.

2. Prove Theorem 2.3, Taylor expansion, multi-index version.

3. Prove Theorem 2.8, tensor as multilinear maps

4. Recall that a function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |a_i - a_{i+1}| < \delta \quad \Longrightarrow \quad \sum_{i=1}^{n} |f(a_i) - f(a_{i+1})| < \varepsilon,$$

whenever $[a_1, a_2], \cdots, [a_n, a_{n+1}]$ are nonoverlapping subintervals of $[a, b]$.

- Check the construction of the Cantor function $c : [0, 1] \to [0, 1]$
- Is the Cantor function absolutely continuous?
- Show that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous, and if $\varphi : [a, b] \to \mathbb{R}$ is a smooth function that vanishes at the end points of the interval, then

$$\int_a^b f'(x)\varphi(x)dx = -\int_a^b f(x)\varphi'(x)dx. \tag{0.1}$$

5*. Try whether you can prove the following statement: if $f : [a, b] \to \mathbb{R}$ is an integrable function such that

$$\int_a^b g(x)\varphi(x)dx = -\int_a^b f(x)\varphi'(x)dx \tag{0.2}$$

for some integrable $g : [a, b] \to \mathbb{R}$ and for all smooth function $\varphi : [a, b] \to \mathbb{R}$ is a smooth function that vanishes at the end points of the interval, then f agrees almost everywhere with an absolutely continuous function; moreover, $f' = g$ almost everywhere in this case.

Hint: You are able to use the following fact: Let $f \in L^1(I)$ be such that

$$\int_I f\varphi' dx = 0$$

for all $\varphi \in C_0^1(I)$. Show that there exists a constant C such that $f = c$ a.e. on I.

Proof. Consider the function $\bar{u}(t) := \int_a^t g(s)ds$. Since $g \in L^1([a, b])$, $\bar{u}$ is absolutely continuous on $[a, b]$ and for all $\varphi \in C_0^\infty([a, b])$

$$\begin{aligned}\int_a^b \bar{u}(x)\varphi'(x)dx &= -\int_a^b g(x)\varphi(x) \\ &= \int_a^b f(x)\varphi'(x)dx.\end{aligned}$$

Thus
$$\int_a^b (f - \bar{u})\varphi'(x)dx = 0$$
for all $\varphi \in C_0^1([a, b])$. By the fact in the hint, we have for some constant $C \in \mathbb{R}$,
$$f - \bar{u} \equiv C \quad \text{a.e. on } [a, b].$$
Set $\bar{f} = \bar{u} + C$. Then $f = \bar{f}$ a.e. on $[a, b]$, $\bar{f}$ is an absolutely continuous function on $[a, b]$, and $f' = g = \bar{f}'$ a.e. on $[a, b]$. □