Exercise Two

1. Prove Lemma 2.17, properties about exterior derivatives in two and three dimensions.

2. Suppose B_1 and B_2 are two concentric balls in $\mathbb{R}^n$ with $\overline{B_1} \subset B_2$. Show that there exists a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that

- $0 \leq f \leq 1$
- $f \equiv 1$ on B_1 and $f = 0$ outside B_2.

3. Suppose U and V are two nonempty open sets in $\mathbb{R}^n$ and $\overline{V}$ is compact with $\overline{V} \subset U$. Show that there exists a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that

- $0 \leq f \leq 1$
- $f \equiv 1$ on V and $f = 0$ outside U.

4. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = (f_1(x,y), f_2(x,y)) := \Big(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\Big).$$

Show that $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$. Is there any smooth function $F\colon \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial x} = f_1 \quad \text{and} \quad \frac{\partial F}{\partial y} = f_2?$$

5*. Prove the Poincaré Lemma 2.22 for $n = 3$.