AN INTRODUCTION TO RIEMANNIAN GEOMETRY

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Contents

1. Introduction

Question 1.1. What kinds of quantities and operations appear in relation to analysis (or multivariable calculus) in a bounded open set $U \subset \mathbb{R}^n$?

Some possible answers:

- Functions: continuity, partial derivatives, integrals, L^p spaces, Taylor expansions, Fourier or related expansions
- Vector fields: gradient, curl, divergence
- Measures, distributions, flows
- Laplace operator, Laplace, heat and wave equations
- Integration by parts formulas (Gauss, divergence, Green)
- Tensor fields, differential forms
- Distance, distance-minimizing curves (line segments), area, volume, perimeter

Imagine similar concepts on a hypersurface (e.g. double torus in $\mathbb{R}^3$)

This course is an introduction to analysis on manifolds. The first part of the course title has the following Wikipedia description: "Mathematical Analysis is a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a

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definition of nearness (a topological space) or specific distances between objects (a metric space)."

Following this description, our purpose will be to study in particular differentiation, integration, and differential equations on spaces that are more general than the standard Euclidean space $\mathbb{R}^n$. Different classes of spaces allow for different kinds of analysis:

- *Topological spaces* are a good setting for studying continuous functions and limits, but in general they do not have enough structure to allow studying derivatives
- The smaller class of *metric spaces* admits certain notions of differentiability, but in particular higher order derivatives are not always well defined
- *Differentiable manifolds* are modeled after pieces of Euclidean space and allow differentiation and integration, but they do not have a canonical Laplace operator and thus the theory of differential equations is limited

The class of spaces studied in this course will be that of Riemannian manifolds. These are differentiable manifolds with an extra bit of structure, a Riemannian metric, that allows to measure lengths and angles of tangent vectors. Adding this extra structure leads to a very rich theory where many different parts of mathematics come together. We mention a few related aspects, and some of these will be covered during this course (the more advanced topics that will be covered will be chosen according to the interests of the audience):

(1) *Calculus.* Riemannian manifolds are differentiable manifolds, hence the usual notions of multivariable calculus on differentiable manifolds apply (derivatives, vector and tensor fields, integration of differential forms)
(2) *Metric geometry.* Riemannian manifolds are metric spaces: there is a natural distance function on any Riemannian manifold such that the corresponding metric space topology coincides with the usual topology. Distances are realized by certain distinguished curves called geodesics, and these can be studied via a second order ODE (the geodesic equation).
(3) *Measure theory.* Any oriented Riemannian manifold has a canonical measure given by the volume form. The presence of this measure allows to integrate functions and to define L^p spaces on Riemannian manifolds.
(4) *Differential equations.* There is a canonical Laplace operator on any Riemannian manifold, and all the classical linear partial differential equations (Laplace, heat, wave) have natural counterparts
(5) *Dynamical systems.* The geodesic flow on a closed Riemannian manifold is a Hamiltonian flow on the cotangent bundle, and the geometry of the manifold is reflected in properties of the flow (such as complete integrability or ergodicity)
(6) *Conformal geometry.* The notions of conformal and quasiconformal mappings make sense on Riemannian manifolds, and there is enough underlying structure to provide many tools for studying them
(7) *Topology.* There are several ways of describing topological properties of the underlying manifold in terms of analysis. In particular, Hodge theory characterizes the cohomology of the space via the Laplace operator acting on differential forms,

and Morse theory describes the topological type of the space via critical points of a smooth function on it

(8) *Curvature.* The notion of curvature is fundamental in mathematics, and Riemannian manifolds are perhaps the most natural setting for studying curvature. Related concepts include the Riemann tensor, the Ricci tensor, and scalar curvature. There has been recent interest in lower bounds for Ricci curvature and their applications
(9) *Inverse problems.* Many interesting inverse problems have natural formulations on Riemannian manifolds, such as integral geometry problems where one tries to determine a function from its integrals over geodesics, or spectral rigidity problems where one tries to determine properties of the underlying space from knowledge of eigenvalues of the Laplacian.
(10) *Geometric analysis.* There are many branches of mathematics that are called geometric analysis. One particular topic is that of geometric evolution equations, where geometric quantities evolve according to a certain PDE. One of the most famous such equations is Ricci flow, where a Riemannian metric is deformed via its Ricci tensor. This was recently used by Perelman to complete Hamilton's program for proving the Poincaré and geometrization conjectures.

2. Calculus in Euclidean spaces

Let U be any nonempty open subset of R^n (not necessarily bounded, and could be equal to R^n). We fix standard Cartesian coordinates $x = (x_1, \cdots, x_n)$ and will use these coordinates throughout this chapter. We may sometimes write x^j instead of x_j, and we will also denote by v_j or v^j the j-th coordinate of a vector $v \in \mathbb{R}^n$.

2.1. Functions and Taylor expansions.

Let $C(U)$ be the set of continuous functions on U. For partial derivatives, we will write

$$\partial_j f = \frac{\partial f}{\partial x_j} \quad \text{and} \quad \partial_{j_1 \cdots j_k} f = \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}.$$

We denote by $C^k(U)$ the set of k times continuously differentiable real valued functions on U. Thus

$$C^k(U) = \Big\{ f \colon U \to \mathbb{R} : \partial_{j_1 \cdots j_l} f \in C(U) \text{ whenever } l \leq k \text{ and } j_1, \cdots, j_l \in \{1, \cdots, n\} \Big\}.$$

Recall also that if $f \in C^k(U)$, then $\partial_{j_1 \cdots j_k} f = \partial_{j_{\sigma(1)} \cdots j_{\sigma(k)}}$ for any permutation σ of $\{1, \cdots, k\}$.

We also denote by $C^\infty(U)$ the infinitely differentiable functions on U, that is,

$$C^\infty(U) = \bigcap_{k \geq 0} C^k(U).$$

Theorem 2.1 (Taylor expansion)**.** Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then

$$f(x) = \sum_{l=0}^{k} \frac{1}{l!} \Big[\sum_{j_1, \cdots, j_l} \partial_{j_1 \cdots j_l} f(x_0)(x - x_0)_{j_1} \cdots (x - x_0)_{j_l} \Big] + R_k(x; x_0),$$

where $|R_k(x;x_0)| \le \eta(|x-x_0|)|x-x_0|^k$ for some function η with $\eta(s)\to 0$ as $s\to 0$.

Remark 2.2. The Taylor expansion of order 2 is given by

$$f(x)=f(x_0)+\nabla f(x_0)\cdot(x-x_0)+\frac{1}{2}\nabla^2 f(x_0)(x-x_0)\cdots(x-x_0)+R_2(x;x_0),$$

where $\nabla f=(\partial_1 f,\cdots,\partial_n f)$ is the gradient of f and $\nabla^2 f(x)=\big(\partial_{jk}f(x)\big)_{j,k=1}^n$ is the Hessian matrix of f.

Proof. Considering $g(y):=f(x_0+y)$, we may assume that $x_0=0$. Assume that $B(0,r)\subset U$, fix $x\in B(0,r)$, and define

$$h\colon(-1-\varepsilon,1+\varepsilon)\to\mathbb{R},\quad h(t):=g(tx),$$

where $\varepsilon>0$ satisfies $(1+\varepsilon)|x|<r$. Then h is a C^k function on $(-1-\varepsilon,1+\varepsilon)$, and repeated use of the fundamental theorem of calculus gives

$$\begin{aligned}
h(t)&=h(t)-h(0)+h(0)=h(0)+\int_0^t h'(s)ds\\
&=h(0)+h'(0)t+\int_0^t(h'(s)-h'(0))ds=h(0)+h'(0)t+\int_0^t\int_0^s h''(u)duds\\
&=h(0)+h'(0)t+h''(0)\frac{t^2}{2}+\int_0^t\int_0^s(h''(u)-h''(0))duds\\
&=\cdots\\
&=\sum_{i=0}^k h^{(i)}(0)\frac{t^i}{i!}+\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\big(h^{(k)}(t_k)-h^{(k)}(0)\big)dt_k\cdots dt_1. \qquad (2.1)
\end{aligned}$$

Here we used that $\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}dt_k\cdots dt_1=\frac{t^k}{k!}$ (exercise).

Now, computation shows

$$h'(t)=\partial_j f(tx)x_j,\quad h''(t)=\partial_{jl}f(tx)x_jx_l,\quad\cdots$$

and

$$h^{(k)}(t)=\partial_{j_1\cdots j_k}f(tx)x_{j_1}\cdots x_{j_k}.$$

Applying (2.1) with $t=1$ gives the result in the theorem, where

$$R_k(x)=\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\Big[\partial_{j_1\cdots j_k}f(t_kx)-\partial_{j_1\cdots j_k}f(0)\Big]x_{j_1}\cdots x_{j_k}dt_k\cdots dt_1.$$

The bound for R_k follows since $\partial_{j_1\cdots j_k}f$ is uniformly continuous on compact sets. $\square$

At this point it may be good to mention another convenient form of the Taylor expansion, which we state but will not use. Let $\mathbb{N}=\{0,1,2,\cdots\}$ be the set of natural numbers. Then $\mathbb{N}^n$ consists of all n-tuples $\alpha=(\alpha_1,\cdots,\alpha_n)$ where the α_j are nonnegative integers. Such an n-tuple is called a multi-index. We write $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. For partial derivatives, the notation

$$\partial^\alpha=\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}$$

will be used. We also use the notation $\alpha! = \alpha_1! \cdots \alpha_n!$.

Theorem 2.3 (Taylor expansion, multi-index version)**.** Let $f \in C^k(U)$, let $x_0 \in U$, and assume that $B(x_0, r) \subset U$. If $x \in B(x_0, r)$, then

$$f(x) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_k(x_0; x),$$

where R_k satisfies similar bounds as before.

Proof. Exercise. □

2.2. **Tensor fields.** If $f \in C^k(U)$, if $x \in U$ and if $v \in \mathbb{R}^n$ is such that $|v|$ is sufficiently small, we write the Taylor expansion given in Theorem 2.1 in the form

$$f(x+v) = \sum_{l=0}^{k} \frac{1}{l!}\Big[\sum_{j_1, \cdots, j_l = 1}^{n} \partial_{j_1 \cdots j_l} f(x) v_{j_1}\Big] + R_k(x+v; x).$$

The first few terms are

$$f(x+v) = f(x) + \partial_j f(x) v_j + \frac{1}{2}\partial_{jk} f(x) v_j v_k + \cdots$$

Looking at the terms of various degree motivates the following definition.

Definition 2.4 (Tensor fields)**.** An *m-tensor field* in U is a collection of functions $u = \big(u_{j_1 \cdots j_m}\big)_{j_1, \cdots, j_m = 1}^{n}$, where each $u_{j_1 \cdots j_m}$ is in $C^\infty(U)$. The tensor field u is called *symmetric* if $u_{j_1 \cdots j_m} = u_{j_{\sigma(1)} \cdots j_{\sigma(m)}}$ for any $j_1, \cdots, j_m$ and for any σ which is a permutation of $\{1, \cdots, m\}$.

Remark 2.5. This definition is specific to R^n, since we are deliberately not allowing any other coordinate systems than the Cartesian one. Later on we will consider tensor fields on manifolds, and their transformation rules under coordinate changes will be an important feature (these will decide whether the tensor field is covariant, contravariant or mixed). However, upon fixing a local coordinate system, all tensor fields will look essentially like the ones defined above.

Example 2.6. (1) The 0-tensor fields in U are just the scalar functions $u \in C^\infty(U)$
(2) The 1-tensor fields in U are of the form $u = (u_j)_{j=1}^n$, where $u_j \in C^\infty(U)$. Thus 1-tensor fields are exactly the vector fields in U; the tensor $(u_j)_{j=1}^n$ is identified with $(u_1, \cdots, u_n)$.
(3) The 2-tensor fields in U are of the form $u = \big(u_{j,k}\big)_{j,k=1}^n$, where $u_{jk} \in C^\infty(U)$. Thus 2-tensor fields can be identified with smooth matrix functions in U. The 2-tensor field is symmetric if the matrix is symmetric.
(4) If $f \in C^\infty(U)$, then we have for any $m \geq 0$ an m-tensor field $u = \big(\partial_{j_1 \cdots j_m} f\big)_{j_1, \cdots, j_m = 1}^n$ consisting of partial derivatives of f. This tensor field is symmetric since the mixed partial derivatives can be taken in any order.

Again by looking at the terms in the Taylor expansion, one can also think that an m-tensor $u = \big(u_{j_1 \cdots j_m}\big)_{j_1, \cdots, j_m = 1}^n$ acts on a vector $v \in \mathbb{R}^n$ by the formula

$$v \mapsto u_{j_1 \cdots j_m}(x) v^{j_1} \cdots v^{j_m}.$$

The last expression can be interpreted as a multilinear map acting on the m-tuple of vectors $(v, \cdots, v)$.

Definition 2.7 (Multi-linear map)**.** If $m \geq 0$, an m-linear map is any map

$$L\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$$

such that L is linear in each of its variables separately.

The following theorem is almost trivial, but for later purposes it will be good to know that a tensor field can be thought of in two ways: either as a collection of coordinate functions, or as a map on U that takes values in the set of multilinear maps.

Theorem 2.8 (Tensors as multilinear maps)**.** If $u = \big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^n$ is an m-tensor field on $U \subset \mathbb{R}^n$, then for any $x \in U$, there is an m-linear map $u(x)$ defined via

$$u(x)(v_1, \cdots, v_m) = u_{j_1\cdots j_m}(x) v_1^{j_1} \cdots v_m^{j_m}, \quad v_1, \cdots, v_m \in \mathbb{R}^n,$$

and it holds that $u_{j_1\cdots j_m}(x) = u(x)(e_{j_1}, \cdots, e_{j_m})$. Conversely, if T is a function that assigns to each $x \in U$ an m-linear map $T(x)$, and if the function $u_{j_1\cdots j_m}\colon x \mapsto T(x)(e_{j_1}, \cdots, e_{j_m})$ are in $C^\infty(U)$ for each $j_1, \cdots, j_m$, then $(u_{j_1\cdots j_m})$ is an m-tensor field in U.

Proof. Exercise. □

2.3. **Vector fields and differential forms.** Let $U \subset \mathbb{R}^n$ be an open set. We wish to consider vector

fields on U and certain operations related to vector fields.

Definition 2.9 (Vector fields)**.** A C^k *vector field* in U is a map $F = (F_1, \cdots, F_n)\colon U \to \mathbb{R}^n$ such that all the component functions F_j are in $C^k(U)$. The set of vector fields on U is denoted by $C^k(U, \mathbb{R}^n)$.

Recall from Section 2.2 that vector fields are the same as 1-tensor fields. If $u \in C^\infty(U)$, the gradient of u gives rise to a vector field in U:

$$\mathrm{grad}\colon C^\infty(U) \to C^\infty(U, \mathbb{R}^n), \quad \mathrm{grad}(u) = (\partial_1 u, \cdots, \partial_n u).$$

If $F \in C^\infty(U, \mathbb{R}^n)$, the *divergence* of F gives rise to a function in U:

$$\mathrm{div}\colon C^\infty(U, \mathbb{R}^n) \to C^\infty(U), \quad \mathrm{div}(F) = \partial_1 F_1 + \cdots + \partial_n F_n.$$

The following basic identity suggests that in order to define the Laplace operator on a space, it may be enough to have a reasonable definition of divergence and gradient.

Lemma 2.10. $\mathrm{div} \circ \mathrm{grad} = \Delta$.

Proof. $\mathrm{div}\big(\mathrm{grad}(u)\big) = \partial_1(\partial_1 u) + \cdots + \partial_n(\partial_n u) = \Delta u.$ □

We will consider further operations on vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Curl in $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be open. If $F \in C^\infty(U, \mathbb{R}^2)$, the curl of F is the function

$$\mathrm{curl}(F) := \partial_1 F_2 - \partial_2 F_1.$$

Thus $\mathrm{curl}\colon C^\infty(U, \mathbb{R}^2) \to C^\infty(U)$.

Curl in $\mathbb{R}^3$. Let $U \subset \mathbb{R}^3$ be open. If $F \in C^\infty(U, \mathbb{R}^3)$, the curl of F is the vector field

$$\operatorname{curl}(F) := \nabla \times F = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1).$$

Lemma 2.11. In two dimensions, one has

$$\operatorname{curl} \circ \operatorname{grad} = 0.$$

In three dimensions, one has

$$\operatorname{curl} \circ \operatorname{grad} = 0, \quad \operatorname{div} \circ \operatorname{curl} = 0.$$

Proof. If $U \subset \mathbb{R}^2$ and $u \in C^\infty(U)$, we have

$$\operatorname{curl}\big(\operatorname{grad}(u)\big) = \partial_1(\partial_2 u) - \partial_2\big(\partial_1 u\big) = 0.$$

If $U \subset \mathbb{R}^3$ and $u \in C^\infty(U)$, we have

$$\operatorname{curl}\big(\operatorname{grad}(u)\big) = \big(\partial_2\partial_3 u - \partial_3\partial_2 u, \partial_3\partial_1 u - \partial_1\partial_3 u, \partial_1\partial_2 u - \partial_2\partial_1 u\big) = 0.$$

Moreover, for $F \in C^\infty(U, \mathbb{R}^3)$ we have

$$\operatorname{div}\big(\operatorname{curl}(F)\big) = \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) = 0.$$

$\square$

The previous lemma can be described in terms of two sequences: if $U \subset \mathbb{R}^2$ consider

$$C^\infty(U) \overset{\text{grad}}{\to} C^\infty(U, \mathbb{R}^2) \overset{\text{curl}}{\to} C^\infty(U) \tag{2.2}$$

and if $U \subset \mathbb{R}^3$ consider

$$C^\infty(U) \overset{\text{grad}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{curl}}{\to} C^\infty(U, \mathbb{R}^3) \overset{\text{div}}{\to} C^\infty(U). \tag{2.3}$$

In both sequences, the composition of any two subsequent operators is zero. This suggests that there may be further structure which underlies these situations and might extend to higher dimensions. This is indeed the case, and the calculus of differential forms (or exterior algebra) was developed to reveal this structure. We will next discuss this calculus in a simple case.

Differential forms. The purpose will be to rewrite for instance (2.3) as a sequence

$$\Omega^0(U) \overset{d}{\to} \Omega^1(U) \overset{d}{\to} \Omega^2(U) \overset{d}{\to} \Omega^3(U), \tag{2.4}$$

where $\Omega^k(U)$ will be the set of differential k-forms on $U \subset \mathbb{R}^3$, and d will be a universal operator that reduces to grad, curl, and div in the respective degrees.

Let $U \subset \mathbb{R}^n$ be open. Motivated by (2.2) and (2.3), we define

$$\Omega^0(U) := C^\infty(U)$$

and

$$\Omega^1(U) := C^\infty(U, \mathbb{R}^n).$$

Thus $\Omega^0(U)$ is the set of smooth functions in U, and any $\alpha \in \Omega^1(U)$ can be identified with a vector field $\alpha = (\alpha_j)_{j=1}^n$, where $\alpha_j \in C^\infty(U)$. We write formally

$$\alpha = (\alpha_j)_{j=1}^n = \alpha_j dx^j.$$

Remark 2.12. For the purposes of this section it is enough to think of dx^j as a formal object. However, the proper way to think of dx^j would be as a 1-form (the exterior derivative of the function $x^j : U \to \mathbb{R}$), i.e. as a map that assigns to each $x \in U$ the linear map $dx^j|_x : T_xM \to \mathbb{R}$ that satisfies $dx^j|_x(e_k) = \delta^j_k$, where $\{e_1, \cdots, e_n\}$ is the standard basis of $T_xM \approx \mathbb{R}^n$.

To define $\Omega^k(U)$ for $k \geq 2$, first define the set of ordered k-tuples

$$\mathcal{I}_k := \{(i_1, \cdots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

If $I \in \mathcal{I}_k$, we consider the formal object

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

Then $\Omega^k(U)$ will be thought of as the set

$$\Omega^k(U) = \{\alpha_I dx^I : \alpha_I \in C^\infty(U)\},$$

where the sum is over all $I \in \mathcal{I}_k$. The number of elements in $\mathcal{I}_k$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We can make the above formal definition rigorous.

Definition 2.13 (Differential form)**.** If $U \subset \mathbb{R}^n$, define for $0 \leq k \leq n$

$$\Omega^k(U) := C^\infty\Big(U, \mathbb{R}^{\binom{n}{k}}\Big).$$

The elements of $\Omega^k(U)$ are called *differential k-forms* on U, and any differential k-form $\alpha \in \Omega^k(U)$ can be written as

$$\alpha = (\alpha_I)_{I \in \mathcal{I}_k} = \alpha_I dx^I,$$

where $\alpha_I \in C^\infty(U)$ for each I.

Remark 2.14. Note that since $\binom{n}{k} = \binom{n}{n-k}$, the set $\Omega^{n-1}(U)$ can be identified with the set of vector fields on U, and $\Omega^n(U)$ with $C^\infty(U)$. In fact one has

$$\Omega^{n-1}(U) = \Big\{ \sum_{j=1}^n \alpha_j dx^1 \wedge \cdots \wedge \hat{dx^j} \wedge \cdots \wedge dx^n; \alpha_j \in C^\infty(U) \Big\}$$

$$\Omega^n(U) = \Big\{ f dx^1 \wedge \cdots \wedge dx^n; f \in C^\infty(U) \Big\},$$

where $\hat{dx^j}$ means that dx^j is omitted from the wedge product.

The above definition is correct, but to keep things simple we have avoided a detailed discussion of the *wedge product* $\wedge$. To define the d operator in (2.4) properly we need to say a little bit more. The wedge product is an associative product on elements of the form dx^I, satisfying

$$dx^j \wedge dx^k = -dx^k \wedge dx^j,$$

and more generally if $J = (j_1, \cdots, j_k)$ is a k-tuple, with $j_1, \cdots, j_k \in \{1, \cdots, n\}$ (not necessarily ordered), we should have

$$dx^{j_1} \wedge \cdots \wedge dx^{j_k} = (-1)^{\text{Sign}(\sigma)} dx^{j_{\sigma(1)}} \wedge \cdots \wedge dx^{j_{\sigma(k)}},$$

where σ is any permutation of $\{1, \cdots, k\}$. This implies two conditions:

- $dx^{j_1} \wedge \cdots \wedge dx^{j_k} = 0$ if $(j_1, \cdots, j_k)$ contains a repeated index
- $dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ can be expressed as $\pm dx^I$ for a unique $I \in \mathcal{I}_k$ if $(j_1, \cdots, j_k)$ contains no repeated index.

With this understanding we make the following definition.

Definition 2.15 (Exterior derivative)**.** The *exterior derivative* is the map $d\colon \Omega^k(U) \to \Omega^{k+1}(U)$ defined by

$$d(\alpha_I dx^I) := \partial_j \alpha_I dx^j \wedge dx^I.$$

Example 2.16. (1) If $f \in \Omega^0(U)$ (so $f \in C^\infty(U)$), then df is the differential of f written as a 1-form:

$$df = \partial_j f dx^j.$$

(2) If $\alpha \in \Omega^1(U)$, say $\alpha = \alpha_k dx^k$ for some $\alpha_j \in C^\infty(U)$, then

$$d\alpha = \partial_j \alpha_k dx^j \wedge dx^k = \sum_{1 \le j < k \le n} (\partial_j \alpha_k - \partial_k \alpha_j) dx^j \wedge dx^k.$$

(3) Any $u \in \Omega^n(U)$ satisfies $du = 0$ since $dx^{j_2} \wedge \cdots \wedge dx^{j_{n+1}} = 0$ whenever $j_1, \cdots, j_{n+1} \in \{1, \cdots, n\}$ and there will be a repeated index.

The second example above gives an n-dimensional analogue of the curl operator, as also suggested by the following lemma:

Lemma 2.17 (The exterior derivatives in two and three dimensions)**.** (1) Let $U \subset \mathbb{R}^2$. If $f \in \Omega^0(U)$, then

$$df = (\operatorname{grad}(f))_j dx^j.$$

If $\alpha = F_1 dx^1 + F_2 dx^2 \in \Omega^1(U)$ and $F = (F_1, F_2)$, then

$$d\alpha = \big(\operatorname{curl}(F)\big) dx^1 \wedge dx^2.$$

(2) Let $U \subset \mathbb{R}^3$. If $f \in \Omega^0(U)$, then

$$df = \big(\operatorname{grad}(f)\big)_j dx^j.$$

If $\alpha = F_j dx^j \in \Omega^1(U)$ and $F = (F_1, F_2, F_3)$, then

$$d\alpha = \big(\operatorname{curl}(F)\big)_j dx^{\hat{j}},$$

where

$$dx^{\hat{1}} := dx^2 \wedge dx^3, dx^{\hat{2}} := dx^3 \wedge dx^1, \text{and } dx^{\hat{3}} := dx^1 \wedge dx^2.$$

Finally, if $u = F_j dx^{\hat{j}} \in \Omega^2(U)$ and $F = (F_1, F_2, F_3)$, then

$$du = \big((\operatorname{div}(F))\big) dx^1 \wedge dx^2 \wedge dx^3.$$

Proof. Exercise. □

Let us now verify that $d \circ d$ is always zero.

Lemma 2.18. $d \circ d = 0$ on $\Omega^k(U)$ for any k with $0 \le k \le n$.

Proof. If $\alpha = \alpha_I dx^I \in \Omega^k(U)$, then

$$d\alpha = \sum_{k=1}^{n} \sum_{I \in \mathcal{I}_k} \partial_k \alpha_I dx^k \wedge dx^I$$

and

$$d(d\alpha) = \sum_{j,k=1}^{n} \sum_{I \in \mathcal{I}_k} \partial_{jk} \alpha_I dx^j \wedge dx^k \wedge dx^I.$$

By the properties of the wedge product, we get

$$d(d\alpha) = \sum_{1 \le j < k \le n} \sum_{I \in \mathcal{I}_k} \big(\partial_{jk}\alpha_I - \partial_{kj}\alpha_I\big) dx^j \wedge dx^k \wedge dx^I,$$

which is zero since the mixed partial derivatives are equal. □

If $U \subset \mathbb{R}^n$ is open, we therefore have a sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U) \tag{2.5}$$

and the composition of any two subsequent operators is zero. This gives the desired generalization of (2.2) and (2.3) to any dimension. In fact we have obtained much more: as we will see during this course, differential forms turn out to be an object of central importance in many kinds of of analysis on manifolds.

Differential forms as tensors. It will be useful to intepret differential forms as tensor fields satisfying an extra condition.

Definition 2.19 (Alternating tensor field)**.** An m-tensor field $\big(u_{j_1\cdots j_m}\big)_{j_1,\cdots,j_m=1}^{n}$ in $U \subset \mathbb{R}^n$ is called *alternating* if $u_{j_{\sigma(1)}\cdots j_{\sigma(m)}} = (-1)^{\text{Sign}(\sigma)} u_{j_1\cdots j_m}$ for any $j_1, \cdots, j_m$ and for any σ which is a permutation of $\{1, \cdots, m\}$.

We understand that 0-tensor fields and 1-tensor fields are always alternating. A 2-tensor field $u = (u_{jk})_{j,k=1}^{n}$ is alternating if and only if $u_{kj} = -u_{jk}$ for any j, k, i.e. the matrix (u_{jk}) is skew-symmetric at each point. An m-tensor field $u = (u_{j_1\cdots j_m})$ is alternating if and only if $u_{j_1\cdots j_m}$ changes sign when any two indices are interchanged (since any permutation can be expressed as the product of transpositions). Note that for an alternating tensor, $u_{j_1\cdots j_m} = 0$ whenever $(j_1, \cdots, j_m)$ contains a repeated index.

Theorem 2.20. If $U \subset \mathbb{R}^n$ is open and $0 \le k \le n$, the set $\Omega^k(U)$ can be identified with the set of alternating k-tensor fields on U.

Proof. Consider the map

$$T \colon \Omega^k(U) \to \{\text{alternating } k\text{-tensors}\}, \quad \alpha dx^I \mapsto \big(\tilde{\alpha}_{j_1\cdots j_k}\big),$$

where

$$\tilde{\alpha}_{j_1\cdots j_k} := \begin{cases} 0, \ (j_1, \cdots, j_k) \text{ contains a repeated index,} \\ \frac{1}{\sqrt{k!}}(-1)^{\text{Sign}(\sigma)}\alpha_I, \ (j_1, \cdots, j_k) \text{ contains no repeated index.} \end{cases}$$

Here, σ is the permutation of $\{1, \cdots, k\}$ such that $I = (j_{\sigma(1)}, \cdots, j_{\sigma(k)})$ is the unique element of $\mathcal{I}_k$ containing the same entries as $(j_1, \cdots, j_k)$. The constant $\frac{1}{\sqrt{k!}}$ is a harmless normalizing factor which will be useful later. Then $\tilde{\alpha}_{j_1\cdots j_k}$ is alternating by construction. It is clear that T is injective, and surjectivity follows since any alternating tensor is uniquely determined by the elements $\tilde{\alpha}_I$ where $I \in \mathcal{I}_k$. $\square$

Cohomology. By Lemma (2.18), we observe that

$$u = d\alpha \text{ for some } \alpha \in \Omega^{k-1}(U) \Rightarrow du = 0.$$

This may be rephrased as follows:

$$\mathrm{Im}\left(d|_{\Omega^{k-1}(U)}\right) \text{ is a linear subspace of } \ker\left(d|_{\Omega^k(U)}\right).$$

We express this in one more way: if $u \in C^k(\Omega)$, we say that u is closed if $du = 0$ and that u is exact if $u = d\alpha$ for some $\alpha \in C^{k-1}(U)$. Thus, any exact differential form is closed. The question of whether any closed form is exact depends on the topological properties of U. To study this property we make the following definition.

Definition 2.21 (de Rham cohomology)**.** The de Rham cohomology groups of U are defined by

$$H^k_{\mathrm{dR}}(U) = \ker\left(d|_{\Omega^k(U)}\right) / \,\mathrm{Im}\left(d|_{\Omega^{k-1}(U)}\right), \quad 0 \leq k \leq n.$$

By this definition each $H^k_{\mathrm{dR}}(U)$ is in fact a (quotient) vector space, not just a group. Any closed k-form is exact if and only if $H^k_{\mathrm{dR}}(U) = \{0\}$. This happens for all $k \geq 1$ at least when U has very simple topology.

Lemma 2.22 (Poincaré lemma)**.** If $U \subset \mathbb{R}^n$ is open and star-shaped with respect to some $x_0 \in U$ (meaning that for any $x \in U$ the line segment between x_0 and x lies in U), then

$$H^k_{\mathrm{dR}}(U) = \begin{cases} \mathbb{R}, & k = 0, \\ \{0\}, & 1 \leq k \leq n. \end{cases}$$

Proof. For simplicity we only do the proof for $n = 2$, see [8] for the general case (which is somewhat more involved). Assume that U is star-shaped with respect to 0. We have

$$H^0_{\mathrm{dR}}(U) = \ker\left(d|_{\Omega^0(U)}\right) = \left\{f \in C^\infty(U), \mathrm{grad}(f) = 0\right\}.$$

Since U is connected and star-shaped with respect to 0, $\nabla f = 0$ on U implies that $f \equiv f(0)$ is constant. Thus $H^0_{\mathrm{dR}}(U)$ is one-dimensional and isomorphic to $\mathbb{R}$.

We next show that $H^1_{\mathrm{dR}}(U) = \{0\}$, that is, for any $F \in C^\infty(U, \mathbb{R}^2)$, we have

$$\mathrm{curl}(F) = 0 \Rightarrow F = \mathrm{grad}(f) \text{ for some } f \in C^\infty(U).$$

Let $F = (F_1, F_2)$ satisfy $\partial_1 F_2 - \partial_2 F_1 = 0$. Then f should be some kind of integral of F, in fact we may just take

$$f(x) := \int_0^1 F_j(tx) x^j \, dt, \quad x \in U.$$

Since $\partial_1 F_2 = \partial_2 F_1$, we have

$$\begin{aligned}\partial_1 f(x) &= \int_0^1 \big[\partial_1 F_j(tx)tx^j + F_1(tx)\big]dt\\ &= \int_0^1 \big[\partial_1 F_1(tx)tx^1 + \partial_2 F_1(tx)tx^2 + F_1(tx)\big]dt\\ &= \int_0^1 \frac{d}{dt}\big[tF_1(tx)\big]dt = F_1(x).\end{aligned}$$

Similarly, $\partial_2 f(x) = F_2(x)$, showing that $F = \mathrm{grad}(f)$.

Finally, we show that $H^2_{\mathrm{dR}}(U) = \{0\}$, which means that

$$f \in C^\infty(U) \Rightarrow f = \mathrm{curl}(F) \quad \text{for some } F \in C^\infty(U, \mathbb{R}^2).$$

As in the previous case, F_j should be integrals of f. We may define

$$F_1(x) := -\int_0^1 f(tx)tx_2 dt \quad \text{and} \quad F_2(x) := \int_0^1 f(tx)tx_1 dx.$$

Then

$$\begin{aligned}\partial_1 F_2 - \partial_2 F_1 &= \int_0^1 \big[\partial_1 f(tx)t^2 x_1 + \partial_2 f(tx)t^2 x_2 + 2tf(tx)\big]dt\\ &= \int_0^1 \frac{d}{dt}\big[t^2 f(tx)\big]dt = f(x).\end{aligned}$$

□

We conclude by mentioning some facts about the de Rham cohomology groups (for more details see [8]):

- The de Rham cohomology groups are *topological invariants*: if U and V are homeomorphic open sets in Euclidean space, then $H^k_{\mathrm{dR}}(U)$ and $H^k_{\mathrm{dR}}(V)$ are isomorphic as vector spaces for each k. This gives a potential way of showing that two sets U and V are not homeomorphic; it would be enough to check that some cohomology groups are not isomorphic
- Note however that it is possible for non-homeomorphic spaces to have the same cohomology groups
- In many cases (e.g. if $U \subset \mathbb{R}^n$ is a bounded open set with nice boundary), the vector spaces $H^k_{\mathrm{dR}}(U)$ are finite dimensional. The dimension of $H^k_{\mathrm{dR}}(U)$ is a known topological invariant, namely the *k-th Betti number* of U.
- Very loosely speaking, the cohomology groups may give some information about "holes" in a set. For instance, if $K_1, \cdots, K_N$ are disjoint closed balls in $\mathbb{R}^n$, then

$$H^k_{\mathrm{dR}}\big(\mathbb{R}^n \backslash \cup_j^N K_j\big) = \begin{cases}\mathbb{R}, \text{if } k = 0,\\ \mathbb{R}^N, \text{if } k = n-1,\\ \{0\}, \text{otherwise}\end{cases}$$

Later in this course we will discuss Hodge theory, which studies the cohomology groups $H^k_{\mathrm{dR}}(M)$ where M is a compact manifold via the Laplace operator acting on differential forms on M.

2.4. **Riemannian metrics.** An open set $U \subset \mathbb{R}^n$ is often thought to be "homogeneous" (the set looks the same near every point) and "flat" (if U is considered as a subset of $\mathbb{R}^{n+1}$ lying in the hyperplane $\{x_{n+1} = 0\}$, then U has the geometry induced by the flat hypersurface $\{x_{n+1} = 0\}$. In this section, we will introduce extra structure on U which makes it "inhomogeneous" (the properties of the set vary from point to point) and "curved" (U has some geometry that is different from the geometry induced by a flat hypersurface $\{x_{n+1} = 0\}$).

Motivation. An intuitive way of introducing this extra structure is to think of U as a medium where sound waves propagate. The properties of the medium are described by a function $c\colon U \to \mathbb{R}_+$, which is thought of as the sound speed of the medium. If U is homogeneous, the sound speed is constant ($c(x) = 1$ for each $x \in U$), but if U is inhomogeneous, then the sound speed varies from point to point.

Consider now a C^1 curve $\gamma\colon [0,1] \to U$. The tangent vector $\dot\gamma(t)$ of this curve is thought to be a vector at the point $\gamma(t)$. If the sound speed is constant ($c \equiv 1$), the length of the tangent vector is just the Euclidean length:

$$\Big|\dot\gamma(t)\Big|_e := \Big[\sum_{j=1}^n \dot\gamma^j(t)^2\Big]^{\frac{1}{2}}.$$

In case of a general sound speed $c\colon U \to \mathbb{R}_+$, one can think that at points where c is large the curve moves very quickly and consequently has short length. Thus we may define the length of $\dot\gamma(t)$ with respect to the sound speed c by

$$\Big|\dot\gamma(t)\Big|_c := \frac{1}{c(\gamma(t))}\Big[\sum_{j=1}^n \dot\gamma^j(t)^2\Big]^{\frac{1}{2}}.$$

It is useful to generalize the above setup in two directions. First, in addition to measuring lengths of tangent vectors we would also like to measure angles between tangent vectors (in particular we want to know when two tangent vectors are orthogonal). Second, if the sound speed is a scalar function on U, then the length of a tangent vector is independent of its direction (the medium is *isotropic*). We wish to allow the medium to be *anisotropic*, which will mean that the sound speed may depend on direction and should be a matrix valued function.

In order to measures lengths and angles of tangent vectors, it is enough to introduce an inner product on the space of tangent vectors at each point. The tangent space is defined as follows:

Definition 2.23 (Tangent space)**.** If $U \subset \mathbb{R}^n$ is open and $x \in U$, the *tangent space* at x is defined as

$$T_xU := \{x\} \times \mathbb{R}^n.$$

The *tangent bundle* of U is the set

$$TU := \bigcup_{x\in U} T_xU.$$

Of course, each T_xU can be identified with $\mathbb{R}^n$ (and we will often do so), and a vector $v \in T_xU$ is written in terms of its coordinates as $v = (v^1, \cdots, v^n)$. Now if $\langle\cdot,\cdot\rangle$ is any inner product on $\mathbb{R}^n$, there is some positive definite symmetric matrix $A = (a_{jk})_{j,k=1}^n$ such that

$$\langle v, w\rangle = Av \cdot w, \quad v, w \in \mathbb{R}^n.$$

(The proof is left as an exercise, hint: take $a_{jk} = \langle e_j, e_k\rangle$) The next definition introduces an inner product on the space of tangent vectors at each point:

Definition 2.24 (Riemannian metric)**.** A *Riemannian metric* on U is a matrix-valued function $g = (g_{jk})_{j,k=1}^n$ such that each g_{jk} is in $C^\infty(U)$, and $(g_{jk}(x))$ is a positive definite symmetric matrix for each $x \in U$. The corresponding *inner product* on T_xU is defined by

$$\langle v, w\rangle_g := g_{jk}(x)v^j w^k, \quad v, w \in T_xU.$$

The *length* of a tangent vector is

$$|v|_g := \langle v, v\rangle_g^{1/2} = (g_{jk}(x)v^j v^k)^{1/2}, \quad v \in T_xU.$$

The *angle* between two tangent vectors $v, w \in T_xU$ is the number $\theta_g(v, w) \in [0, \pi]$ defined by

$$\cos\theta_g(v, w) = \frac{\langle v, w\rangle_g}{|v|_g |w|_g}.$$

We will often drop the subscript and write $\langle\cdot,\cdot\rangle$ or $|\cdot|$ if the metric g is fixed. To connect the above definition to the discussion about sound speeds, a scalar sound speed $c(x)$ corresponds to the Riemannian metric

$$g_{jk}(x) = \frac{1}{c(x)^2}\delta_{jk}.$$

Finally, we introduce some notation that will be very useful.

Notation. If $g = (g_{jk})$ is a Riemannian metric on U, we write

$$(g^{jk})_{j,k=1}^n = g^{-1}$$

for the inverse matrix of $(g_{jk})_{j,k=1}^n$, and

$$|g| = \det(g)$$

for the determinant of the matrix $(g_{jk})_{j,k=1}^n$. In particular, we note that $g_{jk}g^{kl} = \delta_j^l$ for any $j, l = 1, \cdots, n$.

2.5. **Geodesics. Lengths of curves**. Consider an open set U that is equipped with a Riemannian metric g. As we saw above, one can measure lengths of tangent vectors with respect to g, and this makes it possible to measure lengths of curves as well.

Definition 2.25 (Regular curve and its length)**.** A smooth map $\gamma\colon [a, b] \to U$ whose tangent vector $\dot\gamma(t)$ is always nonzero is called a *regular curve*. The *length* of γ is defined by

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| dt.$$

The length of a piecewise regular curve is defined as the sum of lengths of the regular parts.

The *Riemannian distance* between two points $p, q \in U$ is defined by

$$d(p,q) := \inf \Big\{ L(\gamma); \gamma\colon [a,b] \to U \text{ is piecewise regular with } \gamma(a) = p \text{ and } \gamma(b) = q \Big\}.$$

Fact. $L(\gamma)$ is independent of the way the curve γ is parametrized, and that we may always parametrize γ by *arc-length* so that $|\dot{\gamma}(t)| = 1$ for all t. (Proof is left as an exercise)

The previous exercise shows that we can always reparametrize a piecewise regular curve γ by arc length, so that one will have $|\dot{\gamma}(t)| = 1$ for all t. A curve satisfying $|\dot{\gamma}(t)| \equiv 1$ is called a *unit speed curve* (similarly a curve satisfying $|\dot{\gamma}(t)| \equiv$constant is called a *constant speed curve*).

Geodesic equation. We now wish to show that any length minimizing curve satisfies a certain ordinary differential equation.

Theorem 2.26 (Length minimizing curves are geodesics)**.** Suppose $U \subset \mathbb{R}^n$ is open, let g be a Riemannian metric on U, and let $\gamma\colon [a,b] \to U$ be a piecewise regular unit speed curve. Assume that γ minimizes the distance between its endpoints, in the sense that

$$L(\gamma) \leq L(\eta)$$

for any piecewise regular curve η from $\gamma(a)$ to $\gamma(b)$. Then γ is a regular curve, and it satisfies the geodesic equation

$$\ddot{\gamma}^l(t) + \Gamma^l_{jk}\big(\gamma(t)\big)\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0, \quad 1 \leq l \leq n, \tag{2.6}$$

where Γ^l_{jk} are the Christoffel symbols of the metric g:

$$\Gamma^l_{jk} := \frac{1}{2} g^{lm}\big(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}\big), \quad 1 \leq j,k,l \leq n.$$

Example 2.27. If g is the Euclidean metric on U, so that $g_{jk}(x) = \delta_{jk}$, then all the Christoffel symbols Γ^l_{jk} are zero. The geodesic equation becomes just

$$\ddot{\gamma}^l(t) = 0, \quad 1 \leq l \leq n.$$

Solving this equation shows that

$$\gamma(t) = tv + w$$

for some vectors $v, w \in \mathbb{R}^n$. Thus Theorem 2.26 recovers the classical fact that any length minimizing curve in Euclidean space is a line segment.

Any smooth curve that satisfies the geodesic equation (2.6) is called a *geodesic*, and the conclusion of Theorem 2.26 can be rephrased so that any length minimizing curve is a geodesic. The fact that length minimizing curves satisfy the geodesic equation gives powerful tools for studying these curves. For instance, one can show that

- any geodesic has constant speed and is therefore regular

- given any $x \in U$ and $v \in T_x U$, there is a unique geodesic starting at point x in direction v
- any geodesic minimizes length at least locally (but not always globally)
- a set U with Riemannian metric g is *geodesically complete*, meaning that every geodesic is defined for all $t \in \mathbb{R}$, if and only if the metric space (U, d_g) is complete (this is the *Hopf-Rinow theorem*).

Variations of curves. Let $\gamma\colon [a,b] \to U$ be a piecewise regular length minimizing curve. We will prove Theorem 2.26 by considering families of curves (γ_s) where $s \in (-\varepsilon, \varepsilon)$ and $\gamma_0 = \gamma$, and all curves γ_s start at $\gamma(a)$ and end at $\gamma(b)$. Such a family is called a *variation* (or a *fixed-endpoint variation*) of γ. By the length minimizing property,

$$L(\gamma_0) \leq L(\gamma_s) \quad \text{for } s \in (-\varepsilon, \varepsilon),$$

so if the dependence on s is at least C^1 we obtain that $\frac{d}{ds} L(\gamma_s)|_{s=0} = 0$. This fact, applied to many different families γ_s, will imply that γ is smooth and solves the geodesic equation.

If (γ_s) is a family of curves with $\gamma_0 = \gamma$, we think of $V(t) := \frac{\partial}{\partial s}\gamma_s(t)|_{s=0}$ as the "infinitesimal variation" of the curve γ that leads to the family (γ_s). The vector $V(t)$ should be thought of as an element of $T_{\gamma(t)}U$. The next result shows that one can reverse this process, and obtain a variation of γ from any given infinitesimal variation V.

In this result and below, we assume that the piecewise regular curve γ is fixed and that there is a subdivision of $[a,b]$,

$$a = t_0 < t_1 < \cdots < t_N < t_{N+1} = b,$$

such that the curves $\gamma|_{(t_j, t_{j+1})}$ is regular for each j with $0 \leq j \leq N$.

Lemma 2.28 (Variations of curves)**.** If $V\colon [a,b] \to \mathbb{R}^n$ is a continuous map such that $V|_{(t_j,t_{j+1})}$ is C^∞ for each j and $V(a) = V(b) = 0$, then there exists $\varepsilon > 0$ and a continuous map

$$\Gamma\colon (-\varepsilon, \varepsilon) \times [a,b] \to U$$

such that the curves $\gamma_s\colon [a,b] \to U$, $\gamma_s(t) := \Gamma(s,t)$ satisfying the following

- each γ_s is a piecewise regular curve with endpoints $\gamma(a)$ and $\gamma(b)$, and $\gamma_s|_{(t_j,t_{j+1})}$ is regular for each j,
- $\gamma_0 = \gamma$,
- $s \mapsto \gamma_s(t)$ is C^∞ and $\frac{d}{ds}\gamma_s(t)|_{s=0} = V(t)$ for each $t \in [a,b]$.

Proof. Define

$$\Gamma\colon (-\varepsilon, \varepsilon) \times [a,b] \to U, \quad \Gamma(s,t) := \gamma(t) + sV(t),$$

where ε is so small that Γ takes values in U. The properties follow immediately from the definition. ☐

We can now compute the derivative $\frac{d}{ds} L(\gamma_s)|_{s=0}$ that was mentioned above. In classical terminology, this is called the *first variation* of the length functional.

Lemma 2.29 (First variation formula)**.** Let γ be a piecewise regular unit speed curve, and let (γ_s) be a variation of γ associated with V as in Lemma 2.28. Then

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = -\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \langle D_t\dot{\gamma}(t), V(t)\rangle dt - \sum_{j=1}^{N}\langle \Delta\dot{\gamma}(t_j), V(t_j)\rangle,$$

where $D_t\dot{\gamma}(t)$ is the element of $T_{\gamma(t)}U$ defined by

$$\big(D_t\dot{\gamma}(t)\big)^l := \ddot{\gamma}^l(t) + \Gamma_{jk}^l\big(\gamma(t)\big)\dot{\gamma}^j(t)\dot{\gamma}^k(t), \quad 1 \leq l \leq n,$$

and $\Delta\dot{\gamma}(t_j) := \dot{\gamma}(t_j+) - \dot{\gamma}(t_j-)$ is the jump of $\dot{\gamma}(t)$ at t_j.

Remark 2.30. We will later give an invariant meaning to $D_t\dot{\gamma}(t)$ and interpret it as the covariant derivative of $\dot{\gamma}(t)$ along the curve γ. However, at this point it is enough to think of $D_t\dot{\gamma}(t)$ just as some expression that comes out when we compute the derivative $\frac{d}{ds}L(\gamma_s)|_{s=0}$.

Proof. Define

$$I(s) := L(\gamma_s) = \sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \Big[g_{pq}\big(\gamma_s(t)\big)\dot{\gamma}_s^p(t)\dot{\gamma}_s^q(t)\Big]^{\frac{1}{2}} dt.$$

To prepare for computing the derivative $I'(0)$, define two vector fields

$$T(t) := \partial_t\gamma_s(t)|_{s=0} = \dot{\gamma}(t), \ \ V(t) := \partial_s\gamma_s(t)|_{s=0}.$$

Since $|\dot{\gamma}_0(t)| = |T(t)| \equiv 1$ and (g_{jk}) is symmetric, we have

$$I'(0) = \frac{1}{2}\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \Big(\partial_r g_{pq}(\gamma(t))V^r(t)T^p(t)T^q(t) + 2g_{pq}(\gamma(t))\dot{V}^p(t)T^q(t)\Big)dt.$$

Integrating by parts in the last term, this shows that

$$\begin{aligned} I'(0) = \sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} &\Big[\frac{1}{2}\partial_r g_{pq}(\gamma)T^pT^q - \partial_m g_{rq}(\gamma)\dot{T}^q\Big]V^r dt \\ &+ \sum_{j=0}^{N}\Big[\langle V(t_{j+1}), T(t_{j+1})\rangle - \langle V(t_j), T(t_j)\rangle\Big]. \end{aligned}$$

Using that $V(t_0) = V(t_{N+1}) = 0$ and that V is continuous, the boundary term becomes $-\sum_{j=1}^{N}\langle \Delta\dot{\gamma}(t_j), V(t_j)\rangle$ as required. For the integrals, we use that

$$\partial_m g_{rq}(\gamma)T^mT^q = \frac{1}{2}\Big(\partial_m g_{rq}(\gamma) + \partial_q g_{rm}(\gamma)\Big)T^mT^q,$$

which gives

$$\begin{aligned} -\langle D_t\dot{\gamma}(t), V(t)\rangle &= -g_{rq}(\gamma)\big(\dot{T}^q + \Gamma_{jk}^q T^jT^k\big)V^r \\ &= -g_{rq}(\gamma)\Big(\dot{T}^q - \frac{1}{2}\big[\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}\big]T^jT^k\Big)V^r \\ &= -g_{rq}(\gamma)\Big(\dot{T}^q + \frac{1}{2}\partial_r g_{pq}T^pT^q - \partial_m g_{rq}T^mT^q\Big)V^r. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.26. Let $\gamma\colon [a,b] \to U$ be a piecewise regular unit speed curve that minimizes the length between its endpoints. If V is any vector field as in Lemma 2.28 and (γ_s) is the corresponding variation of γ, we must have

$$L(\gamma_0) \le L(\gamma_s)$$

for $s \in (-\varepsilon, \varepsilon)$. Therefore, $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$. The first variation formula, Lemma 2.29, then shows that

$$\sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} \langle D_t\dot{\gamma}(t), V(t)\rangle dt + \sum_{j=1}^{N} \langle \Delta\dot{\gamma}(t_j), V(t_j)\rangle = 0 \tag{2.7}$$

for any such V.

We first show that γ solves the geodesic equation on each interval (t_j, t_{j+1}). Fix $j \in \{0, \cdots, N\}$ and choose V such that

$$V(t) := \varphi(t) D_t\dot{\varphi}(t),$$

where φ is any function in $C_0^\infty\big((t_j, t_{j+1})\big)$. This V is an an admissible choice in Lemma 2.29 and (2.7) implies that □

The previous proof shows actually more than stated in the theorem. We say that a piecewise regular curve γ is a *critical point* of the length functional L if $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$ for any fixed-endpoint variation of γ as in Lemma 2.28.

Theorem 2.31. The critical points of L are exactly the geodesic curves.

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